King Saud University
College of Sciences
Department of Mathematics

Midterm Exam, S3-1444H
ACTU 475
Credibility Theory and Loss Distributions
Time: 2 hours - Marks: 30

Answer the following questions.

Q1: [10]

Determine the survival, density, hazard rate, mean excess loss and limited expected value functions for the age at death model that defined as:

$$F(x)=\begin{cases} 0, & x<0 \\ 0.01x, & 0\le x<100 \\ 1, & x\ge 100 \end{cases}$$

Q2: [5+5]

(a) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and variance of the gamma distribution.

(b) Let Λ have an exponential distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda)=e^{-\lambda x^{\gamma}}$, $x\ge 0$. Determine the unconditional or marginal distribution of X.

Q3: [5+5]

(a) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimate of the parameter θ for an exponential distribution, and for a gamma distribution with $\alpha=2$. Also, find the value of the log-likelihood function in each case.

(b) Let X have a Pareto distribution with parameters α and θ. Let $Y=\ln(1+X/\theta)$. Determine the distribution of Y and its parameters (give its name).

The Model Answer

Q1: [10]

(i) $S(x) = 1 - 0.01x,\ 0 \le x < 100$

(ii) $f(x) = F^{'}(x) = -S^{'}(x) = 0.01$

(iii)

$$\because h(x) = \frac{f(x)}{S(x)}$$
$$\therefore h(x) = \frac{0.01}{1-0.01x}$$

(v)

$$\because e_X(d) = \frac{\int_d^{\infty} S(x)dx}{S(d)}$$
$$= \frac{\int_d^{100} (1-0.01x)dx}{1-0.01d}$$
$$= \frac{-1}{0.01(1-0.01d)}\left[\frac{(1-0.01x)^2}{2}\right]_d^{100}$$
$$\therefore e_X(d) = \frac{1-0.01d}{0.02} = \frac{100-d}{2}$$

(vi)

$$\because E(X \wedge u) = \int_{-\infty}^{u} xf(x)dx + u[1-F(u)]$$
$$= \int_0^u x(0.01)dx + u(1-0.01u)$$
$$= 0.005u^2 + u - 0.01u^2$$
$$\therefore E(X \wedge u) = u(1-0.005u)$$

or

$$E(X \wedge u) = \int_0^u S(x)dx$$

$$= \int_0^u (1-0.01x)dx$$

$$= \left[x - \frac{0.01x^2}{2} \right]_0^u$$

$$\therefore E(X \wedge u) = u - 0.005u^2$$

or

By using the following formula

$$E(X \wedge u) = E(X) - e(u)S(u)$$

$$\because E(X) = \int_0^u x(0.01)dx = \int_0^{100} x(0.01)dx$$

$$= 0.01 \left[\frac{x^2}{2} \right]_0^{100}$$

$$\therefore \text{E(x)} = 50$$

Also, $e(u) = \frac{100-u}{2}$ and $S(u) = 1-0.01u$

$$\because E(X \wedge u) = E(X) - e(u)S(u)$$

$$= 50 - (\frac{100-u}{2})(1 - 0.01u)$$

$$= 50 - \frac{0.01}{2}(100-u)^2$$

$$\therefore E(X \wedge u) = u - 0.005u^2$$

Q2: [5+5]

(a)

For $X \sim gamma(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

$$=\frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^{\alpha}}$$

Where, $r(\theta)=-1/\theta$, $q(\theta)=\theta^{\alpha}$ and $p(x)=x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$\therefore$ The mean, $E(X)=\mu(\theta)=\dfrac{q^{'}(\theta)}{r^{'}(\theta)q(\theta)}$

$$=\frac{\alpha\theta^{\alpha-1}}{1/\theta^{2}.\theta^{\alpha}}=\alpha\theta$$

and the variance, $Var(X)=v(\theta)$

$$=\frac{\mu^{'}(\theta)}{r^{'}(\theta)}$$

$$=\frac{\alpha}{1/\theta^{2}}=\alpha\theta^{2}$$

(b)

$\because X|\Lambda \sim \text{weibull}(\lambda,\gamma),\ \ S_{X|\Lambda}(x|\lambda)=e^{-\lambda x^{\gamma}}$

$\therefore\ A(x)=x^{\gamma}$

$\because \Lambda\ \sim\ \exp\ (\theta)$

$\therefore M_{\Lambda}(z)=(1-\theta z)^{-1}$

$\because\ S_{X}(x)=E[e^{-\Lambda A(x)}]$

$=M_{\Lambda}[-A(x)]$

$\therefore\ S_{X}(x)=M_{\Lambda}[-x^{\gamma}]$

$\therefore\ S_{X}(x)=(1+\theta x^{\gamma})^{-1}$

Which is a loglogistic distribution with the usual parameter θ replaced by $\theta^{\frac{-1}{\gamma}}$.

Q3: [5+5]

For exponential distribution, the likelihood function is

$$
\begin{aligned}
L(\theta) &= f(27)f(82)f(115)f(126)f(155)f(161)f(243) \\
&= \theta^{-1}e^{-27/\theta}\theta^{-1}e^{-82/\theta}\theta^{-1}e^{-115/\theta}\theta^{-1}e^{-126/\theta}\theta^{-1}e^{-155/\theta}\theta^{-1}e^{-161/\theta}\theta^{-1}e^{-243/\theta} \\
&= \theta^{-7}e^{-909/\theta}
\end{aligned}
$$

$$\therefore l(\theta) = -7\ln\theta - 909\theta^{-1}$$

which is known as log-likelihood function, to get the likelihood estimate of the parameter θ

Set $l'(\theta) = 0$

$$\Rightarrow -7\theta^{-1} + 909\theta^{-2} = 0$$

$\therefore \hat{\theta} = 129.85714$ which is the MLE of the mean of an exponential model.

$$\therefore l(\hat{\theta}) = -41.065$$

For a gamma distribution with $\alpha = 2,\ f(x_j|\theta) = x_j\theta^{-2}e^{-x_j/\theta},\ j = 1, 2, \ldots, 7$

$$
\begin{aligned}
\therefore l(\theta) &= \sum_{j=1}^{n}\ln(f(x_j|\theta)) = \sum_{j=1}^{7}\ln(x_j\theta^{-2}e^{-x_j/\theta}) \\
&= \sum_{j=1}^{7}\ln x_j - 14\ln\theta - \theta^{-1}\sum_{j=1}^{7}x_j
\end{aligned}
$$

Set $l'(\theta) = 0$

$$\Rightarrow -14\theta^{-1} + \theta^{-2}\sum_{j=1}^{7}x_j = 0$$

$$-14\theta^{-1} + 7\theta^{-2}\bar{x} = 0 \qquad (\times\frac{\theta^2}{14})$$

$$\therefore \quad \hat{\theta} = \frac{\bar{x}}{2}$$

$$\therefore \quad \hat{\theta} = 64.9286$$

$$\Rightarrow \ l(\hat{\theta}) = \sum_{j=1}^{7}\ln x_j - 14\ln\hat{\theta} - 14$$

$$\therefore l(\hat{\theta}) = -39.5244$$

(b)

$\because X \sim Pareto\ (\alpha, \theta)$

$\therefore F_X(x) = 1 - \left(\frac{\theta}{x+\theta} \right)^{\alpha}$

For $Y = \ln(1 + X / \theta),\ F_Y(y) = \Pr(Y \le y)$

$F_Y(y) = \Pr[\ln(1 + X / \theta) \le y]$

$$\begin{aligned} \therefore F_Y(y) &= \Pr[1 + X / \theta \le e^y] \\ &= \Pr[X \le \theta(e^y - 1)] \\ &= 1 - \left[\frac{\theta}{\theta(e^y - 1) + \theta} \right]^{\alpha} \end{aligned}$$

$$\begin{aligned} \therefore F_Y(y) &= 1 - \left(\frac{1}{e^y} \right)^{\alpha} \\ &= 1 - e^{-\alpha y}, \end{aligned}$$

which is the distribution function of the exponential distribution with parameter $1/\alpha$.
