King Saud University
College of Sciences
Department of Mathematics

Second Midterm Exam, S1-1445H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

Q1: [4+4]

(a) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distribution of X.

(b) Let X have pdf $f(x) = \dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$

Find the pdf as $\theta \to \infty$, $\alpha \to \infty$, and $\theta/\alpha^{1/\gamma} \to \xi$, a constant.

Q2: [4+6]

There are two types of drivers. Good drivers make up 70% of the population and in one year have zero claims with probability 0.6, one claim with probability 0.3, and two claims with probability 0.1. Bad drivers make up the other 30% of the population and have zero, one, or two claims with probabilities 0.4, 0.3, and 0.3, respectively.

(a) Describe this process by using the concept of the risk parameter Θ.

(b) For a particular policyholder, suppose that we have observed $x_1 = 0$ and $x_2 = 1$ for past claims. Determine each of the following:

(i) The posterior distribution of $\Theta | X_1 = 0, X_2 = 1$

(ii) The predictive distribution of $X_3 | X_1 = 0, \text{X}_2 = 1$

Q3: [4+3]

(a) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table

Year	1	2	3
Claims	475	550	400

Determine the full credibility and partial credibility through premium by assuming the normal approximation. Use $r = 0.05$ and $p = 0.95$.

(b) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimate of the parameter θ for **inverse** exponential distribution, and find the value of the log-likelihood function.

Hint: $f(x;\theta) = \dfrac{\theta e^{-\theta/x}}{x^2}$ for inverse exponential distribution.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+4]

(a)

let $\Lambda \sim$ gamma (θ,α), $X|\Lambda \sim$ weibull(λ,γ)

$\therefore \; S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore \; A(x) = x^{\gamma}$

$\because \; S_X(x) = E[e^{-\Lambda A(x)}]$

$\qquad = M_{\Lambda}[-A(x)]$

$\therefore \; S_X(x) = M_{\Lambda}[-x^{\gamma}]$

and $\because M_{\Lambda}(z) = (1-\theta z)^{-\alpha}$

$\therefore \; S_X(x) = (1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with parameters

$\theta \to \theta^{\frac{-1}{\gamma}}, \; \alpha \to \alpha$

(b)

$$\because \; f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad \text{(transformed beta pdf)} \quad (1)$$

let $\theta = \xi\alpha^{1/\gamma}$ where $\theta/\alpha^{1/\gamma} = \xi =$ constant

When $\theta \to \infty$, $\alpha \to \infty$ and $\theta/\alpha^{1/\gamma} \to \xi$, a constant

By using Stirling's formula

$$\lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \qquad (2)$$

$\Rightarrow$

$$\lim_{\alpha+\tau\to\infty} \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}}{\Gamma(\alpha+\tau)} = 1 \qquad (3)$$

Substitute (2) and (3) in (1)

$$f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}\Gamma(\tau)(\xi\alpha^{1/\gamma})^{\gamma\tau}(1+x^{\gamma}\xi^{-\gamma}\alpha^{-1})^{\alpha+\tau}}$$

where $\theta = \xi\alpha^{1/\gamma} \Rightarrow \theta^{-\gamma} = \xi^{-\gamma}\alpha^{-1}$

$$\therefore f(x) = \frac{e^{-\tau}(\frac{\alpha+\tau}{\alpha})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\tau)\xi^{\gamma\tau}(1+\frac{(x/\xi)^{\gamma}}{\alpha})^{\alpha+\tau}}$$

$\because \lim_{a\to\infty}(\frac{\alpha+\tau}{\alpha})^{\alpha+\tau-1/2} = \lim_{\alpha\to\infty}(1+\frac{\tau}{\alpha})^{\alpha+\tau-1/2} = e^{\tau}$

where $\lim_{\alpha\to\infty}(1+\frac{\tau}{\alpha})^{\tau-1/2} = 1$, $\lim_{\alpha\to\infty}(1+\frac{\tau}{\alpha})^{\alpha} = e^{\tau}$

and $\lim_{\alpha\to\infty}(1+\frac{(x/\xi)^{\gamma}}{\alpha})^{\alpha+\tau} = e^{(x/\xi)^{\gamma}}$

$$\therefore \lim_{\alpha\to\infty} f(x) = \frac{\gamma x^{\gamma\tau-1} e^{-(x/\xi)^{\gamma}}}{\Gamma(\tau)\xi^{\gamma\tau}}$$

which is the pdf of the transformed gamma distribution with parameters τ, ξ and γ.

Q2: [4+6]

(a)

x	$\Pr(X = x \mid \Theta = G)$	$\Pr(X = x \mid \Theta = B)$	θ	$\Pr(\Theta = \theta)$
0	0.6	0.4	G	0.7
1	0.3	0.3	B	0.3
2	0.1	0.3		

(b)

(i)

For the posterior distribution, the posterior probabilities are given by

$$\pi(G \mid 0,1) = \frac{f(0 \mid G) f(1 \mid G)\pi(G)}{f_X(0,1)}$$

where $f_X(0,1) = \sum_{\theta} f_{X_1|\Theta}(0 \mid \theta) f_{X_2|\Theta}(1 \mid \theta)\pi(\theta)$

$$f_X(0,1) = 0.6(0.3)(0.7) + 0.4(0.3)(0.3)$$
$$= 0.162$$

$$\pi(G \mid 0,1) = \frac{0.6(0.3)(0.7)}{0.162} = \frac{7}{9} = 0.7778$$

$$\pi(B \mid 0,1) = \frac{0.4(0.3)(0.3)}{0.162} = \frac{2}{9} = 0.2222$$

(ii)

For the predictive distribution, the predictive probabilities are given by

$$f_{X_3|X}(0 \mid 0,1) = \sum_{\theta} f(0 \mid \theta)\pi(\theta \mid 0,1)$$
$$= f(0 \mid G)\pi(G \mid 0,1) + f(0 \mid B)\pi(B \mid 0,1)$$
$$= 0.6(0.7778) + 0.4(0.2222)$$
$$= 0.55556,$$

$$f_{X_3|X}(1 \mid 0,1) = \sum_{\theta} f(1 \mid \theta)\pi(\theta \mid 0,1)$$
$$= f(1 \mid G)\pi(G \mid 0,1) + f(1 \mid B)\pi(B \mid 0,1)$$
$$= 0.3(0.7778) + 0.3(0.2222)$$
$$= 0.3,$$

and $f_{X_3|X}(2 \mid 0,1) = \sum_{\theta} f(2 \mid \theta)\pi(\theta \mid 0,1)$

$$= f(2 \mid G)\pi(G \mid 0,1) + f(2 \mid B)\pi(B \mid 0,1)$$
$$= 0.1(0.7778) + 0.3(0.2222)$$
$$= 0.14444.$$

Q3: [4+3]

(a)

at $p = 0.95,\ \Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

The mean is $\xi = E(X_j) = \dfrac{475+550+400}{3} = 475,$

variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)^2}{n-1} = \dfrac{0^2 + 75^2 + 75^2}{2} = 5625$

For full credibility $n \geq \lambda_0 \left(\dfrac{\sigma}{\xi}\right)^2$

$\therefore n \geq 1536.64 \left(\dfrac{5625}{475^2}\right)$

$\therefore n \geq 38.3095845$

The credibility factor is $Z = \sqrt{\dfrac{n}{\lambda_0 \sigma^2 / \xi^2}}$

$$= \sqrt{\frac{3}{38.3095845}} = 0.279838$$

The partial credibility through premium is

$P_c = Z\overline{X} + (1-Z)M$

$\quad = 0.279838(475) + (1 - 0.279838)(600)$

$\therefore P_c = 565.02025$

(b)

The likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \frac{\theta e^{-\theta/x_j}}{(x_j)^2}$$

The log-likelihood function is

$$l(\theta) = n \ln\theta - \theta \sum_{j=1}^{n} x_j^{-1} - 2\sum_{j=1}^{n} \ln x_j$$

$\therefore l(\theta) = n\ln\theta - ny\theta - 2\sum_{j=1}^{n} \ln x_j$, where $y = \dfrac{1}{n}\sum_{j=1}^{n} x_j^{-1}$

To get the maximum estimate of θ (i.e. $\hat{\theta}$), let $l'(\theta) = 0$

$\Rightarrow l'(\theta) = n\theta^{-1} - ny = 0$

$\Rightarrow \hat{\theta} = \dfrac{1}{y} \ \therefore \hat{\theta} = \dfrac{1}{0.0118} \simeq 84.7$

$$\Rightarrow l(\theta) = n \ln \theta - ny\left(\frac{1}{y}\right) - 2 \sum_{j=1}^{n} \ln x_j$$

$$= n \ln \theta - n - 2 \sum_{j=1}^{n} \ln x_j$$

$$\therefore l(\hat{\theta}) = 7 \ln(84.7) - 7 - 2(32.9) \simeq -41.7$$