King Saud University
College of Sciences
Department of Mathematics

Second Midterm Exam, S2-1445H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

(Note that SND Table is attached in page 2)

Q1: [5+4]

(a) One hundred observed claims in 2020 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 5% per year. Based on the empirical distribution from 2020, determine a range for the probability that a claim exceeds 500 in 2023.

(b) Consider a frailty model with frailty random variable Λ, such that $a(x)=\dfrac{1}{x+1},\ \ x>0.$ Find the conditional survival function of X.

Q2: [4+4]

(a) Let X have pdf $f(x)=\dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$ (Transformed Beta)

Find the pdf as $\theta\to\infty$, $\alpha\to\infty$, and $\theta/\alpha^{1/\gamma}\to\xi$, a constant. Identify the name of this distribution.

(b) Suppose, you have observed the following five claim severities:

16.0, 20.2, 20.0, 19.0 and 36.8. Determine the maximum likelihood estimate of μ for the following model.

$$f(x)=\frac{1}{\sqrt{2\pi x}}\exp[-\tfrac{1}{2x}(x-\mu)^2],\ \ x,\mu>0$$

Q3: [4+4]

Suppose that there were 10 observations of claims with five being zero and others being 253, 398, 439, 129, 627. Let the manual premium $M=220$ and assume that the number of claims has a Poisson distribution. Determine the full credibility and partial credibility in each of the following cases.

(a) According to the average number of claims.

(b) According to the average total payment.

Use $r=0.05$ and $p=0.95$.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [5+4]

(a)

The claim amount in 2020	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 5% per year

In 2021 $\rightarrow$ 1.05 X, in 2022$\rightarrow$ 1.1025 X and in 2023 $\rightarrow$ 1.157625 X

Where, X is the random variable of the claim amount in 2020 and Y=1.157625 X is the random variable of the claim amount in 2023. $Pr(Y > 500) = Pr(X > 500/1.157625) = Pr(X > 431.9)$

From given data, $Pr(X > 400)$ =50/100=0.5 and $Pr(X > 450)$ =45/100=0.45

$\therefore$ $0.45<\Pr(Y > 500) < 0.50$

(b)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\,dt$$

$$= \int_0^x \frac{dt}{1+t} = \ln(1+x)$$

Thus,

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$

$$= e^{-\lambda \ln(1+x)}$$

$$= \frac{1}{(1+x)^{\lambda}}$$

Q2: [4+4]

(a)

$$\because\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad \text{(transformed beta pdf)} \quad (1)$$

let $\theta = \xi\alpha^{1/\gamma}$ where $\theta/\alpha^{1/\gamma} = \xi = \text{constant}$

When $\theta \to \infty,\ \alpha \to \infty$ and $\theta/\alpha^{1/\gamma} \to \xi,$ a constant

By using Stirling's formula

$$\lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \qquad (2)$$

$\Rightarrow$

$$\lim_{\alpha+\tau\to\infty} \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}}{\Gamma(\alpha+\tau)} = 1 \qquad (3)$$

Substitute (2) and (3) in (1)

$$f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}\Gamma(\tau)(\xi\alpha^{1/\gamma})^{\gamma\tau}(1+x^{\gamma}\xi^{-\gamma}\alpha^{-1})^{\alpha+\tau}}$$

where $\theta = \xi\alpha^{1/\gamma} \Rightarrow \theta^{-\gamma} = \xi^{-\gamma}\alpha^{-1}$

$$\therefore f(x)=\frac{e^{-\tau}(\frac{\alpha+\tau}{\alpha})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\tau)\xi^{\gamma\tau}(1+\frac{(x/\xi)^{\gamma}}{\alpha})^{\alpha+\tau}}$$

$$\because \lim_{\alpha\to\infty}(\tfrac{\alpha+\tau}{\alpha})^{\alpha+\tau-1/2}=\lim_{\alpha\to\infty}(1+\tfrac{\tau}{\alpha})^{\alpha+\tau-1/2}=e^{\tau}$$

where $\lim_{\alpha\to\infty}(1+\frac{\tau}{\alpha})^{\tau-1/2}=1,\ \lim_{\alpha\to\infty}(1+\frac{\tau}{\alpha})^{\alpha}=e^{\tau}$

and $\lim_{\alpha\to\infty}(1+\frac{(x/\xi)^{\gamma}}{\alpha})^{\alpha+\tau}=e^{(x/\xi)^{\gamma}}$

$$\therefore \lim_{\alpha\to\infty}f(x)=\frac{\gamma x^{\gamma\tau-1}e^{-(x/\xi)^{\gamma}}}{\Gamma(\tau)\xi^{\gamma\tau}}$$

which is the pdf of the transformed gamma distribution with parameters τ, ξ and γ.

(b)

$$f(x)=\frac{1}{\sqrt{2\pi x}}\exp[-\tfrac{1}{2x}(x-\mu)^2],\ x,\mu>0$$

$$\Rightarrow \ln f(x|\mu)=\ln\frac{1}{\sqrt{2\pi}}+\ln\frac{1}{\sqrt{x}}-\frac{1}{2x}(x-\mu)^2$$

$$=-\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln x-\frac{1}{2}x+\mu-\frac{\mu^2}{2x}$$

$$\because\ l(\mu)=\sum_{j=1}^{n}\ln f_{X_j}(x_j|\mu)$$

$$\therefore\ l(\mu)=-\frac{n}{2}\ln 2\pi-\frac{1}{2}\sum_{j=1}^{n}\ln x_j-\frac{1}{2}\sum_{j=1}^{n}x_j+\mu n-\frac{\mu^2}{2}\sum_{j=1}^{n}\frac{1}{x_j}$$

To get $\hat{\mu}$, set $l'(\mu)=0$

$$\therefore\ l'(\mu)=n-\mu\sum_{j=1}^{n}\frac{1}{x_j}=0$$

$$\Rightarrow\ l'(\mu)=n-n\mu y=0,\ y=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$\therefore\ \hat{\mu}=\frac{1}{y},\ y=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$y=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$\therefore\ \hat{\mu}=\frac{n}{\sum_{j=1}^{n}\frac{1}{x_j}}$$

$$=20.6774$$

Q3: [4+4]

(a)

1. For full credibility

Let N_j be the number of claims where $N_j\sim Poisson(\lambda)$

$E(N_j)=\lambda,\ \sigma^2=Var(N_j)=\lambda$

at $p=0.95,\ \Phi(y_p)=(1+p)/2=0.975$

$\Rightarrow y_p=1.96$ (by using SND table)

$\Rightarrow \lambda_0=(y_p/r)^2=(1.96/0.05)^2=1536.64$

$$n\geq\lambda_0\left(\frac{\sigma}{\xi}\right)^2$$

$$n\geq\lambda_0\left(\frac{\lambda}{\lambda^2}\right)$$

$$n\geq\left(\frac{\lambda_0}{\lambda}\right)$$

$$\therefore n\geq\left(\frac{1536.64}{0.5}\right)$$

$\therefore$ For full credibility $n\geq 3073.28$

2. For partial credibility

The credibility factor is $Z=\sqrt{\frac{10}{3073.28}}=0.057043$

The partial credibility through premium is

$$P_c=Z\overline{X}+(1-Z)M$$

$$=0.057043(184.6)+(1-0.057043)(220)$$

$\therefore P_c=217.98$

(b)

1. For full credibility

$n\geq\frac{\lambda_0}{\lambda}\left[1+\left(\frac{\sigma_Y}{\theta_Y}\right)^2\right]$, where $\lambda_0=1536.64$ and $\lambda=0.5$

The mean is $\theta_Y=\frac{253+398+439+129+627}{5}=369.2,$

variance is $\sigma_Y^2=\frac{\sum_j(y_j-369.2)^2}{4}=35840.2$

$$n\geq\frac{1536.64}{0.5}\left[1+\frac{35840.2}{(369.2)^2}\right]$$

$$n\geq 3073.28\left[1+\frac{35840.2}{(369.2)^2}\right]$$

$\therefore$ For full credibility $n\geq 3881.35$

2. For partial credibility

The credibility factor is $Z=\sqrt{\frac{10}{3881.35}}=0.050758$

The partial credibility through premium is

$$P_c = Z\overline{X} + (1-Z)M$$

$$= 0.050758(184.6) + (1-0.050758)(220)$$

$$\therefore P_c = 218.2$$