King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S1-1443H
ACTU 475 - Credibility Theory and Loss Distributions. - Time: 2 hours

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Answer the following questions:

Q1: [5+4]

(a) One hundred observed claims in 1995 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 10% per year. Based on the empirical distribution from 1995, determine a range for the probability that a claim exceeds 500 in 1998.

(b) Twenty losses are observed. Seven of the losses are 27, 82, 115, 126, 155, 161 and 243. All that is known about the other 13 losses is that they exceed 250. Determine the maximum likelihood estimate of the mean of an exponential model and the value of the log-likelihood function.

Q2: [4+4]

(a) Let X have cdf $F_X(x)=\dfrac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$. Determine the inverse distribution of X, with clarifying, the names of distributions.

(b) Show that the gamma distribution is a member of the exponential family, then derive the mean and the variance of the gamma distribution.

Q3: [4+4]

(a) Let N have a poisson distribution with mean Λ. let Λ have a gamma distribution with mean 1 and variance 2. Determine the unconditional or marginal probability that $N=1$.

(b) Let X have pdf $f(x)=\dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^\gamma]^{\alpha+\tau}}$

Find the pdf as $\theta\to\infty,\ \alpha\to\infty$, and $\theta/\alpha^{1/\gamma}\to\xi,$ a constant.

The Model Answer

Q1: [5+4]

(a)

The amount in 1995	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 10% per year

In 1996 $\rightarrow$ 1.1 X, in 1997 $\rightarrow$ 1.21 X and in 1998 $\rightarrow$ 1.331 X

where X is the random variable of the claim in 1995 and Y=1.331 X is the random variable of the claim in 1998.

$\Pr(Y > 500) = \Pr(X > 500/1.331) = \Pr(X > 376)$

From given data, $\Pr(X > 350)$ =55/100=0.55 and $\Pr(X > 400)$ =50/100=0.50

$\therefore\ 0.50 < \Pr(Y > 500) < 0.55$

(b)

$$
\begin{aligned}
L(\theta) &= f(27)f(82)f(115)f(126)f(155)f(161)f(243).[S(250)]^{495} \\
&= \theta^{-1}e^{-27/\theta}\theta^{-1}e^{-82/\theta}\theta^{-1}e^{-115/\theta}\theta^{-1}e^{-126/\theta}\theta^{-1}e^{-155/\theta}\theta^{-1}e^{-161/\theta}\theta^{-1}e^{-243/\theta}[e^{-250/\theta}]^{13} \\
&= \theta^{-7}e^{-909/\theta}[e^{-250/\theta}]^{13} = \theta^{-7}e^{-4159/\theta}
\end{aligned}
$$

$\therefore\ l(\theta) = -7\ln\theta - 4159\theta^{-1}$

Which is known as log-likelihood function, to get the likelihood estimate of the parameter θ

Set $l'(\theta) = 0$

$\Rightarrow -7\theta^{-1} + 4159\theta^{-2} = 0$

$\therefore \hat{\theta} = 594.14$ which is the MLE of the mean of an exponential model.

$\therefore l(\hat{\theta}) = -51.7098$

Q2: [4+4]

(a)

For loglogistic distribution with 2 parameters γ and θ, $F_X(x) = \dfrac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$

For $Y = X^{-1}$, we have $F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \frac{(y^{-1}/\theta)^\gamma}{1+(y^{-1}/\theta)^\gamma}$$

$$= \frac{1}{1+(y^{-1}/\theta)^\gamma}$$

$$= \frac{(y\theta)^\gamma}{1+(y\theta)^\gamma}$$

Which is also loglogistic distribution with 2 parameters γ and θ, γ unchanged and $\theta \to 1/\theta$.

(b)

For $X \sim gamma(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

$$= \frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^\alpha}$$

Where, $r(\theta) = -1/\theta$, $\text{q}(\theta)=\theta^\alpha$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$\therefore$ The mean, $E(X) = \mu(\theta) = \dfrac{q'(\theta)}{r'(\theta)q(\theta)}$

$$= \frac{\alpha\theta^{\alpha-1}}{1/\theta^2.\theta^{\alpha}} = \alpha\theta$$

and the variance, $Var(X) = v(\theta)$

$$= \frac{\mu^{'}(\theta)}{r^{'}(\theta)}$$

$$= \frac{\alpha}{1/\theta^2} = \alpha\theta^2$$

Q3: [4+4]

(a)

$$\Lambda \sim gamma(\alpha,\theta), \quad N\big|A \sim poisson(\lambda)$$
$$E(\Lambda) = \alpha\theta = 1, \ \mathrm{Var}(\Lambda) = \alpha\theta^2 = 2$$
$$\Rightarrow \theta = 2, \ \alpha = 1/2$$

$$\because f_N(n) = \int f_{N|\Lambda}(n\big|\lambda) f(\lambda) d\lambda$$

$$\therefore f_N(1) = \int_0^{\infty} \frac{e^{-\lambda}\lambda^1}{1!} . \frac{\lambda^{-0.5}e^{-0.5\lambda}}{\Gamma(0.5)2^{0.5}}$$

$$\therefore f_N(1) = \frac{1}{\sqrt{2}\ \Gamma(0.5)} \int_0^{\infty} \lambda^{0.5} e^{-1.5\lambda} d\lambda$$

Let $y = 1.5\lambda \Rightarrow \lambda = \frac{y}{1.5}, \ d\lambda = \frac{dy}{1.5}$

$$\therefore f_N(1) = \frac{1}{1.5\sqrt{3}\ \Gamma(0.5)} \int_0^{\infty} y^{0.5} e^{-y} dy$$
$$= \frac{\Gamma(1.5)}{1.5\sqrt{3}\ \Gamma(0.5)}$$
$$= \frac{0.5\Gamma(0.5)}{1.5\sqrt{3}\ \Gamma(0.5)}$$
$$\therefore f_N(1) = 0.1925$$

(b)

$$\because\ f(x)=\frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}\quad \text{(transformed beta pdf)}\quad (1)$$

let $\theta = \xi\alpha^{1/\gamma}$ where $\theta/\alpha^{1/\gamma} = \xi = \text{constant}$

When $\theta \to \infty,\ \alpha \to \infty$ and $\theta/\alpha^{1/\gamma} \to \xi,$ a constant

By using Stirling's formula

$$\lim_{\alpha\to\infty}\frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)}=1 \qquad (2)$$

$\Rightarrow$

$$\lim_{\alpha+\tau\to\infty}\frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}}{\Gamma(\alpha+\tau)}=1 \qquad (3)$$

Substitute (2) and (3) in (1)

$$f(x)=\frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}\Gamma(\tau)(\xi\alpha^{1/\gamma})^{\gamma\tau}(1+x^{\gamma}\xi^{-\gamma}\alpha^{-1})^{\alpha+\tau}}$$

where $\theta = \xi\alpha^{1/\gamma} \Rightarrow \theta^{-\gamma} = \xi^{-\gamma}\alpha^{-1}$

$$\therefore f(x)=\frac{e^{-\tau}(\frac{\alpha+\tau}{\alpha})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\tau)\xi^{\gamma\tau}(1+\frac{(x/\xi)^{\gamma}}{\alpha})^{\alpha+\tau}}$$

$$\because \lim_{\alpha\to\infty}(\tfrac{\alpha+\tau}{\alpha})^{\alpha+\tau-1/2}=\lim_{\alpha\to\infty}(1+\tfrac{\tau}{\alpha})^{\alpha+\tau-1/2}=e^{\tau}$$

where $\lim_{\alpha\to\infty}(1+\tfrac{\tau}{\alpha})^{\tau-1/2}=1,\ \lim_{\alpha\to\infty}(1+\tfrac{\tau}{\alpha})^{\alpha}=e^{\tau}$

and $\lim_{\alpha\to\infty}(1+\tfrac{(x/\xi)^{\gamma}}{\alpha})^{\alpha+\tau}=e^{(x/\xi)^{\gamma}}$

$$\therefore \lim_{\alpha\to\infty}f(x)=\frac{\gamma x^{\gamma\tau-1}e^{-(x/\xi)^{\gamma}}}{\Gamma(\tau)\xi^{\gamma\tau}}$$

which is the pdf of the transformed gamma distribution with parameters $\tau,\ \xi$ and γ.