King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S2-1442H
ACTU 475
Credibility Theory and Loss Distributions
Time: 2 hours - Marks: 25

Answer the following questions:

(Note that SND Table is attached in page 3)

Q1: [5+4]

(a) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table.

Year	1	2	3
Claims	400	550	475

Determine the standard for full credibility and then determine the net premium for next year's claims assuming the normal approximation. Use $r = 0.05$ and $p = 0.95$.

(b) Five hundred losses are observed. Five of the losses are 1100, 3200, 3300, 3500, and 3900. All that is known about the other 495 losses is that they exceed 4000. Determine the maximum likelihood estimate of the mean of an exponential model and the value of the loglikelihood function.

Q2: [4+4]

(a) Suppose, you have observed the following five claim severities:

16.0, 20.2, 20.0, 19.0 and 36.8. Determine the maximum likelihood estimate of μ for the following model.

$$f(x) = \frac{1}{\sqrt{2\pi x}} \exp[-\tfrac{1}{2x}(x-\mu)^2], \ x, \mu > 0$$

(b) Show that the normal distribution is a member of the linear exponential family.

Q3: [4+4]

(a) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distribution of X.

(b) Let X have pdf

$$f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$$

Find the pdf as $\tau \to \infty$.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [5+4]

(a)

at $p = 0.95,\ \Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

The mean is $\xi = E(X_j)$

$$= \frac{400+550+475}{3} = 475$$

and variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)}{n-1}$

$$= \frac{75^2 + 75^2 + 0^2}{2} = 5625$$

$\Rightarrow \sigma = \sqrt{5625} = 75$

For full credibility

$$n \ge \lambda_0 (\tfrac{\sigma}{\xi})^2$$

$$n \ge 1536.64(\tfrac{75}{475})^2$$

$$n \ge 38.3096$$

The credibility factor is $Z = \sqrt{\frac{n}{\lambda_0 \sigma^2/\xi^2}}$

$$= \sqrt{\tfrac{3}{38.3096}} = 0.2798$$

The partial credibility through premium is

$P_c = Z\overline{X} + (1-Z)M$

$= 0.2798(475) + 0.7202(600)$

$= 565.025 \simeq 565$

(b)

$L(\theta) = f(1100) f(3200) f(3300) f(3500) f(3900).[S(4000)]^{495}$

$= \theta^{-1} e^{-1100/\theta} \theta^{-1} e^{-3200/\theta} \theta^{-1} e^{-3300/\theta} \theta^{-1} e^{-3500/\theta} \theta^{-1} e^{-3900/\theta} [e^{-4000/\theta}]^{495}$

$= \theta^{-5} e^{-1995000/\theta}$

$$\Rightarrow l(\theta) = -5\ln\theta - \frac{1995000}{\theta}$$

To get $\hat{\theta}$, set $l'(\theta) = 0$

$$\therefore \frac{\partial l}{\partial \theta} = \frac{-5}{\theta} + \frac{1995000}{\theta^2} = 0$$

$$\Rightarrow \hat{\theta} = \frac{1995000}{5} = 399000$$

and $l(\hat{\theta}) = -69.4836$

Q2: [4+4]

(a)

$$f(x) = \frac{1}{\sqrt{2\pi x}} \exp[-\tfrac{1}{2x}(x-\mu)^2], \; x, \mu > 0$$

$$\Rightarrow \ln f(x|\mu) = \ln\frac{1}{\sqrt{2\pi}} + \ln\frac{1}{\sqrt{x}} - \frac{1}{2x}(x-\mu)^2$$

$$= -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln x - \frac{1}{2}x + \mu - \frac{\mu^2}{2x}$$

$$\because \; l(\mu) = \sum_{j=1}^{n} \ln f_{X_j}(x_j|\mu)$$

$$\therefore \; l(\mu) = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_{j=1}^{n}\ln x_j - \frac{1}{2}\sum_{j=1}^{n} x_j + \mu n - \frac{\mu^2}{2}\sum_{j=1}^{n}\frac{1}{x_j}$$

To get $\hat{\mu}$, set $l'(\mu) = 0$

$$\therefore \; l'(\mu) = n - \mu\sum_{j=1}^{n}\frac{1}{x_j} = 0$$

$$\Rightarrow \; l'(\mu) = n - n\mu y = 0, \; y = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$\therefore \; \hat{\mu} = \frac{1}{y}, \; y = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$\therefore \; \hat{\mu} = 20.6774$$

(b)

For $X \sim N(\theta, \upsilon^2)$

The pdf is given by

$$f(x;\theta) = (2\pi\upsilon)^{\frac{-1}{2}} \exp[\tfrac{-1}{2\upsilon}(x-\theta)^2]$$
$$= (2\pi\upsilon)^{\frac{-1}{2}} \exp[-\tfrac{x^2}{2\upsilon} + \tfrac{\theta}{\upsilon}x - \tfrac{\theta^2}{2\upsilon}]$$

$$f(x;\theta) = \frac{[(2\pi\upsilon)^{\frac{-1}{2}} \exp(-\frac{x^2}{2\upsilon})]\exp(\frac{\theta}{\upsilon}x)}{\exp(\frac{\theta^2}{2\upsilon})}$$

which of the form $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

where $p(x) = [(2\pi\upsilon)^{\frac{-1}{2}} \exp(-\frac{x^2}{2\upsilon})]$, $r(\theta) = \frac{\theta}{\upsilon}$ and $q(\theta) = \exp(\frac{\theta^2}{2\upsilon})$

$\therefore$ The normal distribution is a member of the linear exponential family.

Q3: [4+4]

(a)

let $\Lambda \sim$ gamma (θ, α), $X|\Lambda \sim$ weibull(λ, γ)

$\therefore \quad S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore \quad A(x) = x^{\gamma}$

$\because \quad S_X(x) = E[e^{-\Lambda A(x)}]$
$= M_{\Lambda}[-A(x)]$

$\therefore \quad S_X(x) = M_{\Lambda}[-x^{\gamma}]$

and $\because M_{\Lambda}(z) = (1-\theta z)^{-\alpha}$

$\therefore \quad S_X(x) = (1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with parameters

$\theta \to \theta^{\frac{-1}{\gamma}}, \ \alpha \to \alpha$

(b)

$\because \quad f(x) = \dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$ (transformed beta pdf) let α be constant and $\theta\tau^{1/\gamma} \to \xi$

$\Rightarrow \quad \theta = \xi\tau^{-1/\gamma}$

$$\therefore \; f(x) = \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau}(1/\xi\tau^{1/\gamma})^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$= \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)(\xi\tau^{-1/\gamma})^{\gamma\tau}[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$\therefore \; f(x) = \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}} \qquad (1)$$

$$\because \; \lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \qquad \text{Stirling's formula}$$

$$\Rightarrow \; \Gamma(\tau) = e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2} \text{ as } \tau \to \infty \qquad (2)$$

$$\Rightarrow \; \Gamma(\alpha+\tau) = e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2} \text{ as } \alpha+\tau \to \infty \qquad (3)$$

Substitute (2), (3) in (1)

$$\therefore \; f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)e^{-\tau}(\tau)^{\tau-1/2}(2\pi)^{1/2}\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\Rightarrow f(x) = \frac{e^{-\alpha}(\frac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}x^{-\gamma\tau-\gamma\alpha}}{\Gamma(\alpha)\tau^{-\alpha}\tau^{-\tau}\xi^{\gamma(\tau+\alpha)}\xi^{-\gamma\alpha}x^{-\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\because \; \lim_{a\to\infty}(1+\tfrac{x}{a})^{a+b} = e^{x} \quad \therefore \; \lim_{\tau\to\infty}(\tfrac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2} = \lim_{\tau\to\infty}(1+\tfrac{\alpha}{\tau})^{\tau+\alpha-1/2} = e^{\alpha}$$

where $\lim_{\tau\to\infty}(1+\tfrac{\alpha}{\tau})^{\alpha} = 1$

$$\therefore \; f(x) = \frac{e^{-\alpha}e^{\alpha}\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}(\frac{1}{\tau})^{\alpha+\tau}(\frac{\xi}{x})^{\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore \; f(x) = \frac{\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}[\frac{1}{\tau}(\frac{\xi}{x})^{\gamma}]^{\alpha+\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore f(x) = \frac{\gamma(\xi/x)^{\gamma\alpha}}{\Gamma(\alpha)x} \cdot \frac{1}{\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}}$$

$$\because \; \lim_{\tau\to\infty}\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau} = e^{(\xi/x)^{\gamma}}$$

$$\therefore f(x) = \frac{\gamma(\xi / x)^{\gamma\alpha} e^{-(\xi/x)^{\gamma}}}{\Gamma(\alpha)x} \quad \text{as } \tau \to \infty$$

which is the inverse transformed gamma pdf with parameters α, ξ and γ