King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S2-1442H
ACTU 475 – Credibility Theory and Loss Distributions
Time: 90 minutes

Answer the following questions:

Q1: [3+1+4]

For the model of automobile bodily injury claim that is defined by an insurance company as

$$F(x)=\begin{cases}0, & x<0,\\ 1-\left(\dfrac{2000}{x+2000}\right)^3, & x\geq 0.\end{cases}$$

a) Determine the survival, density, and hazard rate functions.

b) Construct only the graph of the survival function.

c) Determine the mean excess loss and limited expected value functions.

Q2: [8]

Actuaries at an Insurance Services Office, considered a mixture of two Pareto distributions as follows

$$F(x)=1-a\left(\frac{\theta_1}{\theta_1+x}\right)^{\alpha}-(1-a)\left(\frac{\theta_2}{\theta_2+x}\right)^{\alpha+2}$$

Determine the mean and variance of this mixture distribution.

Q3: [3+3+3]

Suppose that X has an exponential distribution. Determine the cdf of the inverse, transformed, and inverse transformed exponential distributions.

The Model Answer

Q1: [3+1+4]

a) The survival function is

$$S(x) = -F(x)$$

$$\therefore\ S(x) = \left(\frac{2000}{x+2000}\right)^3,\ x \ge 0$$

The density function is

$$f(x) = F^{'}(x) = -S^{'}(x)$$

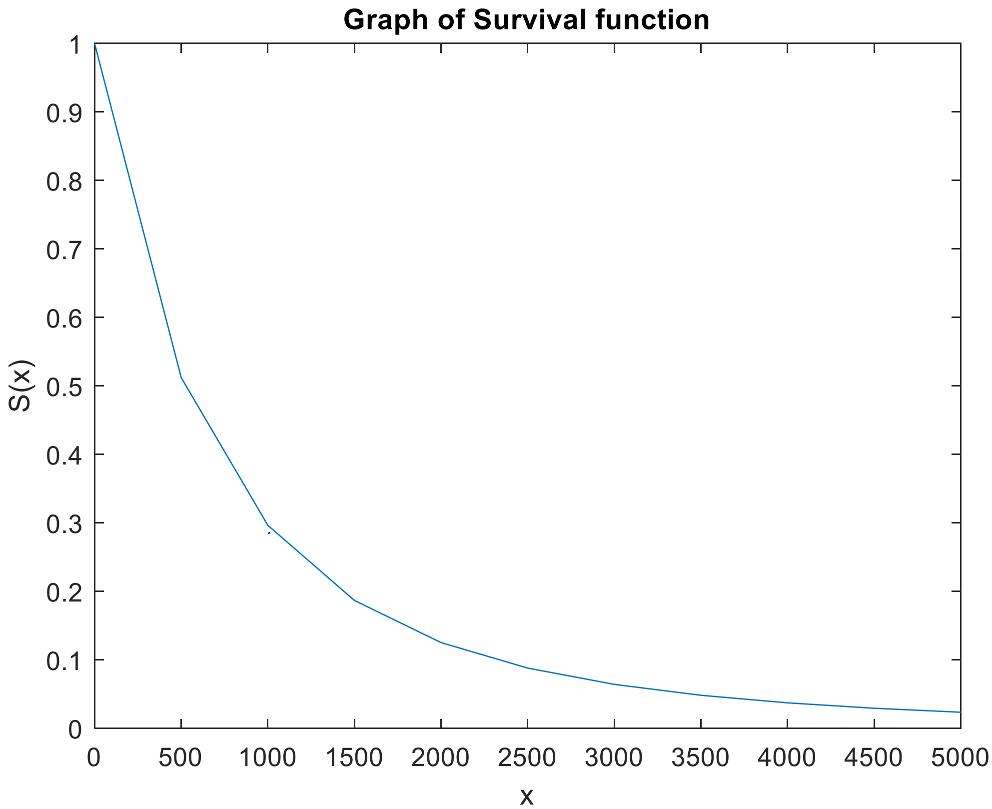
$$\therefore\ f(x) = \frac{3(2000)^3}{(x+2000)^4},\ x > 0$$

The hazard rate function

$$h(x) = \frac{f(x)}{S(x)}$$

$$\therefore\ h(x) = \frac{3}{(x+2000)},\ x > 0$$

b)



c)

The mean excess loss function

$$e(d) = \frac{\int_d^\infty S(x)\ dx}{S(d)}$$

$$= \frac{\int_d^\infty \left(\frac{2000}{x+2000}\right)^3\ dx}{\left(\frac{2000}{d+2000}\right)^3}$$

$$= \frac{[(x+2000)^{-2}]_d^\infty}{-2\left(\frac{1}{d+2000}\right)^3}$$

$$\therefore\ e(d) = \frac{2000+d}{2}$$

To get the limited expected value function $E(X \wedge u)$

First Method

$$E(X \wedge u) = \int_{-\infty}^{u} x f(x) dx + u[1-F(u)]$$

$$\Rightarrow \; E(X \wedge u) = \int_{0}^{u} x \frac{3(2000)^3}{(x+2000)^4} dx + u\left(\frac{2000}{u+2000}\right)^3$$

The integral $I = \int_{0}^{u} x \frac{3(2000)^3}{(x+2000)^4} dx$ could be evaluated to be as follows

$$I = \tfrac{3}{2}(2000)^3 \left[\frac{1}{(2000)^2} - \frac{1}{(u+2000)^2}\right] - (2000)^4 \left[\frac{1}{(2000)^3} - \frac{1}{(u+2000)^3}\right] \; (*)$$

$$\therefore \; E(X \wedge u) = 1000 - \frac{3(2000)^3}{2(u+2000)^2} + \frac{(2000)^4}{(u+2000)^3} + u\left(\frac{2000}{u+2000}\right)^3$$

$$E(X \wedge u) = 1000 - \left(\frac{2000}{u+2000}\right)^3 \left[\tfrac{3}{2}(u+2000) - 2000 - u\right]$$

$$E(X \wedge u) = 1000\left[1 - \frac{4000000}{(u+2000)^2}\right]$$

Second Method

$$E(X \wedge u) = -\int_{-\infty}^{0} F(x) dx + \int_{0}^{u} S(x) dx$$

$$\Rightarrow \; E(X \wedge u) = -\int_{-\infty}^{0} 0 \; dx + \int_{0}^{u} \left(\frac{2000}{x+2000}\right)^3 dx = \int_{0}^{u} \left(\frac{2000}{x+2000}\right)^3 dx$$

$$\therefore \; E(X \wedge u) = (2000)^3 \left[\frac{(x+2000)^{-2}}{-2}\right]_0^u$$

$$E(X \wedge u) = 1000\left[1 - \frac{4000000}{(u+2000)^2}\right]$$

Third Method

By using the following formula

$$E(X \wedge u) = E(X) - e(u)S(u)$$

To obtain $E(X)$, let $u \to \infty$ in (*)

$$\therefore \; E(X) = 1000$$

Also, $e(u)$ is determined before, $e(u) = \frac{2000+u}{2}$

$$E(X \wedge u) = 1000 - \left(\frac{2000+u}{2}\right)\left(\frac{2000}{u+2000}\right)^3$$

$$\therefore\ E(X \wedge u) = 1000\left[1 - \frac{4000000}{(u+2000)^2}\right]$$

Q2: [8]

For the mixture of 2 Pareto distributions

$$F(x) = 1 - a\left(\frac{\theta_1}{\theta_1 + x}\right)^{\alpha} - (1-a)\left(\frac{\theta_2}{\theta_2 + x}\right)^{\alpha+2}$$

The m[th] moment of a k-point mixture distribution is given by

$$\because E(Y^m) = \int y^m [a_1 f_{X_1}(y) + \ldots + a_k f_{X_k}(y)]dy$$

$$\therefore\ E(Y^m) = a_1 E(Y_1^m) + \ldots + a_k E(Y_k^m)$$

For $m = 1$ and two point mixture distribution

$$\Rightarrow\ E(Y) = aE(Y_1) + (1-a)E(Y_2)$$

For Pareto - (α, θ)

$$E(X^k) = \frac{\theta^k k!}{(\alpha-1)\ldots(\alpha-k)}$$, where k is a positive integer

$$\Rightarrow\ E(X) = \frac{\theta}{(\alpha-1)},\ \alpha > 1 \text{ and } E(X^2) = \frac{2\theta^2}{(\alpha-1)(\alpha-2)},\ \alpha > 2$$

$\therefore$ The mean is given by

$$E(Y) = a\frac{\theta_1}{\alpha-1} + (1-a)\frac{\theta_2}{\alpha+2-1}$$

$$E(Y) = a\frac{\theta_1}{\alpha-1} + (1-a)\frac{\theta_2}{\alpha+1},\ \alpha > 1$$

Similarly, for the second moment

$$E(Y^2) = aE(Y_1^2) + (1-a)E(Y_2^2)$$

$$E(Y^2) = a\frac{2\theta_1^2}{(\alpha-1)(\alpha-2)} + (1-a)\frac{2\theta_2^2}{\alpha(\alpha+1)},\ \alpha > 2$$

$$\text{Variance} = E(Y^2) - [E(Y)]^2$$

$$= a\frac{2\theta_1^2}{(\alpha-1)(\alpha-2)} + (1-a)\frac{2\theta_2^2}{\alpha(\alpha+1)} - a^2\frac{\theta_1^2}{(\alpha-1)^2} - (1-a)^2\frac{\theta_2^2}{(\alpha+1)^2} - 2a(1-a)\frac{\theta_1\theta_2}{(\alpha^2-1)}$$

$$\text{Variance} = a\frac{2\theta_1^2}{(\alpha-1)(\alpha-2)} - a^2\frac{\theta_1^2}{(\alpha-1)^2} + (1-a)\frac{2\theta_2^2}{\alpha(\alpha+1)} - (1-a)^2\frac{\theta_2^2}{(\alpha+1)^2} - 2a(1-a)\frac{\theta_1\theta_2}{(\alpha^2-1)}$$

Q3: [3+3+3]

We have, $F_X(x) = 1 - e^{-x}$ (exp, dist. with no scale parameter), so we could obtain the following:

(1) The inverse exponential distribution with no scale parameter (where $\tau = -1$) has cdf

$$F_Y(y) = 1 - F_X(y^{-1}) \qquad \text{Theorem}$$
$$= 1 - [1 - e^{-1/y}]$$
$$F_Y(y) = e^{-1/y}$$

With the scale parameter added, it is $F(y) = e^{-\theta/y}$ (inverse exponential distribution)

(2) The transformed exponential distribution with no scale parameter (where $\tau > 0$) has cdf

$$F_Y(y) = F_X(y^\tau), \qquad \tau > 0$$
$$= 1 - e^{-y^\tau}$$
$$F_Y(y) = 1 - \exp(-y^\tau)$$

With the scale parameter added, it is $F(y) = 1 - \exp[-(y/\theta)^\tau]$ (Weibull distribution)

(3) The inverse transformed exponential distribution with no scale parameter has cdf

$$F_Y(y) = 1 - F_X(y^{-\tau}) \qquad \text{Theorem for negative } \tau$$
$$= 1 - [1 - \exp(-y^{-\tau})]$$
$$F_Y(y) = \exp(-y^{-\tau})$$

With the scale parameter added, it is

$F(y) = \exp[-(y/\theta)^{-\tau}] = \exp[-(\theta/y)^\tau]$ (inverse Weibull distribution)