King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S2-1443H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 90 minutes - Marks: 25

Answer the following questions:

(Note that SND Table is attached in page 2)

Q1: [6+3]

(a) If the random variable X has probability density function $f(x)=(1+2x^2)e^{-2x},\ x\geq 0.$

(i) Determine the survival function.

(ii) Determine the hazard rate function.

(iii) Determine the mean excess loss function.

(b) The cdf of a random variable is $F(x)=1-x^{-2},\ x\geq 1.$ Determine the mean, median, and mode of this random variable.

Q2: [6]

One hundred observed claims in 1995 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 10% per year. Based on the empirical distribution from 1995, determine a range for the probability that a claim exceeds 500 in 1998.

Q3: [5+5]

(a) Seventy-five percent of claims have a normal distribution with a mean of 3,000 and a variance of 1,000,000. The remaining 25% have a normal distribution with a mean of 4,000 and a variance of 1,000,000. Determine the probability that a randomly selected claim exceeds 5,000.

(b) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda)=e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distribution of X.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [6+3]

(a)

(i)

The survival function is

$$S(x)=\int_x^\infty (1+2t^2)e^{-2t}dt$$

$$=-\tfrac{1}{2}e^{-2t}+2I,\ where\ I=\int_x^\infty t^2e^{-2t}dt$$

$$I=\int_x^\infty t^2e^{-2t}dt$$

$$=-\tfrac{1}{4}e^{-2t}-\tfrac{t}{2}e^{-2t}-\tfrac{t^2}{2}e^{-2t}$$

$$\therefore\ S(x)=-(1+t+t^2)e^{-2t}\Big|_x^\infty$$

$$=(1+x+x^2)e^{-2x},\ x\geq 0$$

(ii)

$\because$ The hazard rate function is

$$h(x)=-\frac{d}{dx}[\ln S(x)]$$

and $\because S(x)=(1+x+x^2)e^{-2x}$

$$\ln S(x)=-2x+\ln(1+x+x^2)$$

$$\therefore\ h(x)=2-\frac{1+2x}{1+x+x^2}$$

or

$$h(x)=\frac{f(x)}{S(x)}=\frac{1+2x^2}{1+x+x^2}$$

(iii)

The mean excess loss function is

$$e_X(x)=\frac{\int_x^\infty S(t)dt}{S(x)} \qquad (1)$$

From (i) $S(x)=(1+x+x^2)e^{-2x}$ (2)

$$\int_x^\infty S(t)dt = -(1+t+\tfrac{1}{2}t^2)e^{-2t}\Big|_x^\infty$$
$$= (1+x+\tfrac{1}{2}x^2)e^{-2x} \qquad (3)$$

$\therefore$ By substituting (2) and (3) in (1), we get

$$e_X(x) = \frac{1+x+\frac{1}{2}x^2}{1+x+x^2}$$

(b)

The pdf is $f(x) = 2x^{-3},\ x \ge 1.$

The mean is

$$E(X) = \int_1^\infty 2x^{-2}dx$$
$$= 2$$

To get the median, solve the equation $F(x) = 1 - x^{-2} = 0.5$

$\Rightarrow$ The median $\simeq 1.4142$

The mode is the value at which the pdf is highest, so to get the mode,

$\because\ f(x) = 2x^{-3},\ x \ge 1$ is a decreasing function and its highest value at $x = 1$

$\therefore$ The mode =1

Q2: [6]

The amount in 1995	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 10% per year

In 1996 $\rightarrow$ 1.1 X, in 1997 $\rightarrow$ 1.21 X and in 1998 $\rightarrow$ 1.331 X

where X is the random variable of the claim in 1995 and Y=1.331 X is the random variable of the claim in 1998.

$\Pr(Y > 500) = \Pr(X > 500/1.331) = \Pr(X > 376)$

From given data, $\Pr(X > 350)$ =55/100=0.55 and $\Pr(X > 400)$ =50/100=0.50

$\therefore$ 0.50<$\Pr(Y > 500) < 0.55$

Q3: [5+5]

(a)

For this mixture distribution,

$$
\begin{aligned}
F(5000) &= 0.75\Phi\left(\frac{5000-3000}{1000}\right)+0.25\Phi\left(\frac{5000-4000}{1000}\right)\\
&= 0.75\Phi(2)+0.25\Phi(1)\\
&= 0.75(0.9772)+0.25(0.8413)\\
&= 0.9432
\end{aligned}
$$

$\therefore \Pr(X>5000)=1-0.9432=0.0568,$ where X is a randomly selected claim.

(b)

let $\Lambda \sim$ gamma (θ,α), $X|\Lambda \sim$ weibull(λ,γ)

$\therefore \quad S_{X|\Lambda}(x|\lambda)=e^{-\lambda x^{\gamma}}$

$\therefore \quad A(x)=x^{\gamma}$

$$
\begin{aligned}
\because \quad S_X(x) &= E[e^{-\Lambda A(x)}]\\
&= M_{\Lambda}[-A(x)]
\end{aligned}
$$

$\therefore \quad S_X(x)=M_{\Lambda}[-x^{\gamma}]$

and $\because M_{\Lambda}(z)=(1-\theta z)^{-\alpha}$

$\therefore \quad S_X(x)=(1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with parameters

$\theta \to \theta^{\frac{-1}{\gamma}},\ \alpha \to \alpha$
