King Saud University
College of Sciences
Mathematics Department

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Exercise 1

Consider a permanent disability model with three states: State 0: Healthy, State 1: Permanently disabled, and State 2 : Dead. Suppose that

$$\mu_x^{01} = 0.02,\ \mu_x^{02} = 0.04,\ \mu_x^{12} = 0.05 \text{ for } x \geq 40$$

An insurance company uses the model to calculate premiums for a 5–year term insurance policy issued to a life aged 40 who is now healthy. The death benefit is 100,000, payable at the moment of death. Premiums are payable monthly in advance provided that the policyholder is healthy.

1. *Calculate the* ***monthly premium*** *for this policy on the following basis: Interest: constant force of interest of 6% per year Initial expense: 60% of the gross premium Renewal expenses: 10% of each of the premium except the first. (**Hint** use* ${}_tp_{40}^{00} = e^{-0.06t}$, ${}_tp_{40}^{01} = 2\left(e^{-0.05t} - e^{-0.06t}\right)$ $\ddot{a}_{x:\overline{n}|}^{00}(m) = \frac{1}{m}\sum_{k=0}^{nm-1} e^{-0.06\frac{k}{m}}\ {}_{\frac{k}{m}}p_x^{00}$ *and* $\bar{A}_{x:\overline{n}|}^{02} = \int_0^n e^{-\delta t}\left({}_tp_x^{00}\ \mu_{x+t}^{02} + {}_tp_x^{01}\ \mu_{x+t}^{12}\right)\mathrm{d}t$*).*
2. *Calculate the gross premium reserve at time 2 for a policyholder who is permanently disabled by that time. The reserve basis is the same as the premium basis, with the exception that (i)* $\mu_x^{12} = 0.06$ *for* $x \geq 40$*; (ii) the constant force of interest is 5% per year; (iii) the initial expense is 100% of the gross premium. (**Hint:*** $\bar{A}_{x:\overline{n}|}^{12} = \int_0^n e^{-\delta t}\ {}_tp_x^{11}\ \mu_{x+t}^{12}\ \mathrm{d}t$*.)*

Solution:

1. The transition probabilities have been calculated in the previous question so we have

$$\begin{aligned}
\bar{A}_{40:\overline{5}|}^{02} &= \int_0^5 e^{-0.06t}\left({}_tp_{40}^{00}\ \mu_{40+t}^{02} + {}_tp_{40}^{01}\ \mu_{40+t}^{12}\right)\mathrm{d}t \\
&= \int_0^5 e^{-0.06t}\left(0.04e^{-0.06t} + 2\left(e^{-0.05t} - e^{-0.06t}\right)0.05\right)\mathrm{d}t \\
&= \int_0^5 e^{-0.06t}\left(0.1e^{-0.05t} - 0.06e^{-0.06t}\right)\mathrm{d}t \\
&= \frac{0.1}{0.11}\left(1 - e^{-0.55}\right) - \frac{0.06}{0.12}\left(1 - e^{-0.6}\right) = 0.15899
\end{aligned}$$

and

$$\begin{aligned}
\ddot{a}_{40:\overline{5}|}^{00}(12) &= \frac{1}{12}\sum_{k=0}^{59} e^{-0.06\frac{k}{12}}\ {}_{\frac{k}{12}}p_{40}^{00} = \frac{1}{12}\sum_{k=0}^{59} e^{-0.005k}e^{-0.06\frac{k}{12}} \\
&= \frac{1}{12}\sum_{k=0}^{59} e^{-0.01k} = \frac{1 - e^{-0.6}}{12\left(1 - e^{-0.01}\right)} = 3.77873.
\end{aligned}$$

Let the monthly premium by G. By the equivalence principle,

$$12G\ddot{a}^{00}_{40:\overline{5}|}(12) = 12 \times 0.1G\left(\ddot{a}^{00}_{40:\overline{5}|} - \frac{1}{12}\right) + 0.6G + 100000\bar{A}^{02}_{40:\overline{5}|}$$

$$12 \times 0.9G\ddot{a}^{00}_{40:\overline{5}|} + 0.1G - 0.6G = 100000\bar{A}^{02}_{40:\overline{5}|} \iff 40.31028G = 15899$$

which give $G = \frac{15899}{40.31028} = \mathbf{394.41552}$.

2. Since the life is already disabled and Y can never revert back to state 0, there is no future premium (and hence no renewal expenses). The gross premium reserve is equal to the APV of the death benefit

$$_2V = 100000\bar{A}^{12}_{42:\overline{2}|}$$

where

$$\bar{A}^{12}_{42:\overline{2}|} = \int_0^2 e^{-0.05t}\ {}_tp^{11}_{40}\ \mu^{12}_{40+t}\ \mathrm{d}t = \int_0^2 e^{-0.05t}e^{-0.06t}0.06\ \mathrm{d}t = \frac{0.06}{0.11}\left(1 - e^{-0.22}\right) = 0.10772.$$

So, the gross premium reserve is $_2V = \mathbf{10772}$.

Exercise 2

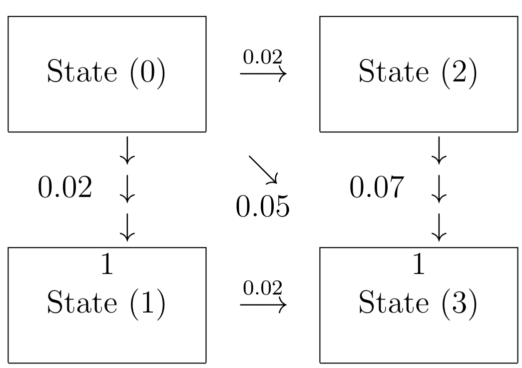
Ahmed and Hajar are considering buying a motorcycle to ride together. If they do not buy the motorcycle, their future lifetimes are independent, each having constant force of mortality of $\mu = 0.02$. If they buy the motorcycle there would be an additional risk that both lives will die simultaneously. This additional risk can be modeled by a common shock with a constant transition intensity of $\gamma = 0.05$. Also, if Ahmed dies before Hajar, Hajar will continue to ride the motorcycle, and will consequently continue to experience the additional force of mortality γ. If Hajar dies before Ahmed, Ahmed will sell the motorcycle, and his mortality will revert to its previous level.

1. *Sketch a diagram that summarizes the states and transitions in the joint life model including the motorcycle–related mortality. You should clearly label each state and transition in the model and show the amount of each transition intensity.*

2. *Ahmed and Hajar purchase a fully continuous joint life insurance policy that pays 50000 on the first death, unless the death arises from a motorcycle accident, in which case the benefit is only 2000. Premiums are payable continuously while both lives survive. Calculate the annual net premium rate for the whole life policy, assuming $\delta = 0.06$. (**Hint:** Let $\bar{A}$ be the APV of 1 unit of benefit payable at the instant when Ahmed and Hajar die together $\bar{A} = \int_0^\infty e^{-\delta t}\ {}_tp^{00}_x\mu^{03}\mathrm{d}t$, $\bar{a}^{00}_{xy} = \int_0^\infty e^{-\delta t}\ {}_tp^{00}_x\mathrm{d}t$, $\bar{A}^{01}_{xy} = \int_0^\infty e^{-\delta t}\ {}_tp^{00}_x\mu^{01}\mathrm{d}t$ and $\bar{A}^{02}_{xy} = \int_0^\infty e^{-\delta t}\ {}_tp^{00}_x\mu^{02}\mathrm{d}t$ where ${}_tp^{00}_x = e^{-t(\mu^{01}+\mu^{02}+\mu^{03})}$).*

Solution:

1. State 0 : Ahmed and Hajar are both alive. State 1: Ahmed is alive, but Hajar is dead. State 2: Ahmed is dead, but Hajar is alive. State 3: Both Ahmed and Hajar are dead.



2. Let $\bar{A}$ be the APV of 1 unit of benefit payable at the instant when Ahmed and Hajar die together. Then the APV of the benefits is

$$\begin{aligned} & 50000\left(\bar{A}_{xy}^{01}+\bar{A}_{xy}^{02}\right)+2000\bar{A} \\ = & 50000\int_0^{\infty} e^{-0.06t}\left(2e^{-0.09t}\times 0.02\right)\mathrm{d}t+2000\int_0^{\infty} e^{-0.06t}\left(e^{-0.09t}\times 0.05\right)\mathrm{d}t=14000 \end{aligned}$$

The APV of premiums is $P\bar{a}_{xy}^{00}=P\frac{1}{0.09+0.06}=\frac{P}{0.15}$. So $P=14000\times 0.15=\mathbf{2100}$.

Exercise 3

Consider a 4–year pure endowment policy sold to (50). The policy pays a survival benefit of \$1000 at the end of the fourth year if the policyholder is alive at that time. Premiums are paid at the beginning of each year as long as the policyholder is alive. You are given: (i) $v(t)$ is the price of a zero–coupon bond that pays \$1 with certainty t years from now. (ii) The following interest rate scenario model:

Scenario	*Probability*	$v(1)$	$v(2)$	$v(3)$	$v(4)$
1	0.3	0.96	0.92	0.88	0.8
2	0.4	0.94	0.88	0.85	0.75
3	0.2	0.99	0.94	0.89	0.85
4	0.1	0.9	0.8	0.7	0.6

(ii) $\mu_x=0.02$ *for all* $x\geq 0$.

1. *Calculate the expected value of the net annual premium.*

2. *Let $f(t,t+k)$ be the forward interest rate, contracted at time 0, effective from time t to $t+k$. Calculate the mean of $f(1,4)$.*

Solution:

1. The net annual premium can be expressed as

$$1000\frac{A_{70:\overline{4}|}}{\ddot{a}_{70:\overline{4}|}}=\frac{1000\nu(4)\ {}_4p_{50}}{1+v(1)\ p_{50}+v(2)\ {}_2p_{50}+\nu(3)\ {}_3p_{50}}$$

Since $\mu_x=0.02$ for all x, we have ${}_tp_{50}=e^{-0.02t}$. The net annual premium $P(\omega)$ for

$$\text{Scenario 1, is } P(\omega_1)=\frac{1000\times 0.8\times e^{-0.08}}{1+0.96e^{-0.02}+0.92e^{-0.04}+0.88e^{-0.06}}=\mathbf{202.12365}$$

$$\text{Scenario 2, is } P(\omega_2)=\frac{1000\times 0.75\times e^{-0.08}}{1+0.94e^{-0.02}+0.88e^{-0.04}+0.85e^{-0.06}}=\mathbf{194.07436}$$

$$\text{Scenario 3, is } P(\omega_3)=\frac{1000\times 0.85\times e^{-0.08}}{1+0.99e^{-0.02}+0.94e^{-0.04}+0.89e^{-0.06}}=\mathbf{211.39827}$$

$$\text{Scenario 4, is } P(\omega_4)=\frac{1000\times 0.6\times e^{-0.08}}{1+0.90e^{-0.02}+0.80e^{-0.04}+0.70e^{-0.06}}=\mathbf{167.32997}$$

The expected value of the net annual premium is given by

$$\begin{aligned} \mathrm{E}[P] &= 0.3P(\omega_1)+0.4P(\omega_2)+0.2P(\omega_3)+0.1P(\omega_4) \\ &= 0.3\times 202.12365+0.4\times 194.07436+0.2\times 211.39827+0.1\times 167.32997=\mathbf{197.27949} \end{aligned}$$

2. We have

$$f(1,4)(\omega) = \left(\frac{v(1)(\omega)}{v(4)(\omega)}\right)^{\frac{1}{3}} - 1$$

The values of $f(1,4)$ for the four interest rate scenarios are

$$\begin{aligned}
\text{Scenario 1, } f(1,4)(\omega_1) &= \left(\frac{0.96}{0.8}\right)^{\frac{1}{3}} - 1 = 0.06266, \\
\text{Scenario 2, } f(1,4)(\omega_2) &= \left(\frac{0.94}{0.75}\right)^{\frac{1}{3}} - 1 = 0.07817 \\
\text{Scenario 3, } f(1,4)(\omega_3) &= \left(\frac{0.99}{0.85}\right)^{\frac{1}{3}} - 1 = 0.05214, \\
\text{Scenario 4, } f(1,4)(\omega_4) &= \left(\frac{0.90}{0.60}\right)^{\frac{1}{3}} - 1 = 0.14471
\end{aligned}$$

The mean of $f(1,4)$ is given by

$$\begin{aligned}
\mathrm{E}[f(1,4)] &= 0.3f(1,4)(\omega_1) + 0.4f(1,4)(\omega_2) + 0.2f(1,4)(\omega_3) + 0.1f(1,4)(\omega_4) \\
&= 0.3 \times 0.06266 + 0.4 \times 0.078174 + 0.2 \times 0.05214 + 0.1 \times 0.14471 = \mathbf{0.07496}.
\end{aligned}$$

Exercise 4

1. *For a fully discrete 2–year life insurance policy on* (40) *you are given: (i) All cash flows are annual. (ii) The annual level gross premium is* 250. *(iii) Profits and premiums are discounted at an annual effective rate of* r. *(iv) The expected profit at the end of each year given that the insurance is in force at the beginning of the year:*

Time in years	0	1	2
Profit	−280	100	230

(v) The profit margin is −8% *and* $q_{40} = 0.004$. *Calculate* r. *(**Hint**: Profit margin* $= \frac{\mathrm{NPV}(r)}{\sum_{k=0}^{n-1} \frac{G\ {}_kp_x}{(1+r)^k}}$, $\mathrm{NPV}(r) = \sum_{k=0}^{n} \frac{\Pi_k}{(1+r)^k}$ *and* $\Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots$ *is the profit signature).*

2. *For a fully discrete 3–year term life insurance policy on* (40) *you are given: (i) All cash flows are annual. (ii) The annual gross premium is* 1000. *(iii) Profits and premiums are discounted at an annual effective rate of* 8%. *(iv) The expected profit at the end of each year given that the insurance is in force at the beginning of the year:*

Time in years	0	1	2	3
Profit Pr_t	−300	140	260	400

(v) The profit signatures are

Time in years	0	1	2	3
Profit signature Π_t	−300	140	245	350

*Calculate the profit margin. (**Hint**: use the hint of 1)*

Solution:

1. We have $p_{40} = 1 - q_{40} = 0.996$. So, the NPV of the premiums is $250(1 + \frac{0.996}{1+r})$. The expected profits given are said to be calculated assuming that the policy is in force at the beginning of the year. So the profit vector is $(-280, 100, 230)'$. The profit signature is

$$(-280, 100, 230 \times 0.996)' = (-280, 100, 229.08)'$$

So, the NPV is

$$-280 + \frac{100}{1+r} + \frac{229.08}{(1+r)^2}.$$

Since the profit margin is -8%

$$-280 + \frac{100}{1+r} + \frac{229.08}{(1+r)^2} = -0.08 \times 250\left(1 + \frac{0.996}{1+r}\right) = -20 - \frac{19.92}{1+r}$$

which gives

$$\frac{229.08}{(1+r)^2} + \frac{119.92}{1+r} - 260 = 0$$

By using a financial calculator, r is found to be $\mathbf{0.19718}$.

2. By comparing (iv) and (v), we get

$$p_{40} = \frac{245}{260} = 0.94231 \text{ and } {}_2p_{40} = \frac{350}{400} = 0.875$$

The NPV is

$$\text{NPV}(0.08) = -300 + \frac{140}{1.08} + \frac{260}{(1.08)^2} + \frac{400}{(1.08)^3} = 370.07062.$$

The APV of premiums is

$$1000\left(1 + \frac{0.94231}{1.08} + \frac{0.875}{(1.08)^2}\right) = 2622.6807.$$

So, the profit margin is $\frac{\text{NPV}(0.08)}{\text{APV of premiums}} = \frac{370.07062}{2622.6807} = \mathbf{14.11}\%$

Exercise 5

1. *For a specified amount universal life insurance policy, you are given: (i) The face amount of the policy is* $100,000 *(ii) The account value on December* 31, 2020 *was* $20,000. *(iii) On January* 1, 2021, *a premium of* $2000 *was made. No other premiums were made in* 2021. *(iv) The expense charge and the cost of insurance deducted on January* 1, 2021 *were* $100 *and* $150, *respectively. (v) The credited interest rate in* 2021 *was* 5% *per annum effective. (vi) The surrender charge applicable in* 2021 *was* $20 *per* $1000 *face amount. Calculate the cash value of the policy on December* 31, 2021. *(**Hint:*** $\text{AV}_t = (\text{AV}_{t-1} + P_t - \text{EC}_t - \text{CoI}_t)(1 + i_t^c)$ *and* $\text{CV}_t = \max(\text{AV}_t - \text{SC}_t, 0)$*).*

2. *Consider a specified amount universal life insurance policy. You are given: (i) The policyholder is exactly aged 40 at the beginning of the year. (ii) The account value at the beginning of the year is* $22,000. *(iii) At the beginning of the year, a premium of* $3,000 *is made. No other premiums are made in the same year. (iv) The death benefit (the face amount) is* $120,000, *payable at the end of the year of death. (v) The expense charge is a flat amount* $50 *plus* 3% *of each premium.*

*(vi) The corridor factor for the year is 2.3. The following basis is used to calculate the cost of insurance: - Interest: 3% per annum effective and $q_t^* = q_{40} = 0.003$. Assume that the **credited interest rate** for the year is 1% per annum effective. Calculate the cost of insurance deducted at the beginning of the year. (**Hint**: If the corridor factor requirement is satisfied, then*

$$\mathrm{CoI}_t^f = \frac{q_t^*\ v_q\left(\mathrm{FA} - \left(\mathrm{AV}_{t-1} + P_t - \mathrm{EC}_t\right)\left(1 + i_t^c\right)\right)}{1 - q_t^*\ v_q\left(1 + i_t^c\right)}$$

If the corridor requirement is not satisfied, then

$$\mathrm{CoI}_t^c = \frac{q_t^*\ v_q\left(1 + i_t^c\right)\left(\gamma_t - 1\right)\left(\mathrm{AV}_{t-1} + P_t - \mathrm{EC}_t\right)}{1 + q_t^* v_q\left(1 + i_t^c\right)\left(\gamma_t - 1\right)}.)$$

Solution:

1. First, we need to project the account value on December 31, 2021. The account value on December 31, 2021 is given by

$$\mathrm{AV}_t = \left(\mathrm{AV}_{t-1} + P_t - \mathrm{EC}_t - \mathrm{CoI}_t\right)\left(1 + i_t^c\right) = (20000 + 2000 - 100 - 150)(1.05) = 22837.5$$

The cash value of the policy on December 31, 2021 is given by

$$\mathrm{CV}_t = \max\left(\mathrm{AV}_t - \mathrm{SC}_t, 0\right) = \max(22837.5 - 2000, 0) = \mathbf{20837.5}$$

2. If the corridor factor requirement is satisfied, then fro $t = 31/12/2021$ and $t - 1 = 31/12/2020$

$$\mathrm{CoI}_t^f = \frac{q_t^*\ v_q\left(\mathrm{FA} - \left(\mathrm{AV}_{t-1} + P_t - \mathrm{EC}_t\right)\left(1 + i_t^c\right)\right)}{1 - q_t^*\ v_q\left(1 + i_t^c\right)}$$

We are given $v_q = (1+0.03)^{-1}$, $\mathrm{FA} = 120000$, $\mathrm{AV}_{t-1} = 22000$, $P_t = 3000$, $\mathrm{EC}_t = 50 + 0.03 \times 3000 = 140$ and $i_t^c = 0.01$. We have $q_t^* = q_{40} = 0.003$. This gives

$$\mathrm{CoI}_t^f = \frac{0.003\,(1.03)^{-1}\,(120000 - (22000 + 3000 - 140)(1.01))}{1 - 0.003\,(1.03)^{-1}\,(1.01)} = 277.19817$$

If the corridor requirement is not satisfied, then

$$\begin{aligned}\mathrm{CoI}_t^c &= \frac{q_t^*\ v_q\left(1 + i_t^c\right)\left(\gamma_t - 1\right)\left(\mathrm{AV}_{t-1} + P_t - \mathrm{EC}_t\right)}{1 + q_t^*\ v_q\left(1 + i_t^c\right)\left(\gamma_t - 1\right)} \\ &= \frac{0.003\,(1.03)^{-1}\,(1.01)(2.3 - 1)(22000 + 3000 - 140)}{1 + 0.003\,(1.03)^{-1}\,(1.01)(2.3 - 1)} = 94.7092\end{aligned}$$

Hence, the cost of insurance is $\mathrm{CoI}_t = \max(\mathrm{CoI}_t^f,\ \mathrm{CoI}_t^c) = \max(277.19817, 94.7092) = \mathbf{277.19817}$.