King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2019–2020
Academic Year (H) 1441
Bachelor AFM: M. Eddahbi

Solution of the first midterm Spring 2020 ACTU. 462 (25%) (two pages)

March 1, 2020 (two hours 3–5 PM)

Problem 1. (6 marks)

1. For a special fully continuous whole life insurance on (65): (i) The death benefit at time t is $b_t = 2000e^{0.05t}$, for $t \geq 0$, (ii) Level premiums are payable for life, (iii) $\mu_{65+t} = 0.04$, $t \geq 0$ and $\delta = 0.05$

   (a) **(2 marks)** Calculate the annual net premium rate for this life insurance.

   (b) **(2 marks)** Calculate the premium reserve at the end of year 2.

2. **(2 marks)** For a fully discrete whole life insurance of 1000 on (50), you are given:
   $1000P_{50} = 25$, $1000A_{61} = 440$, $1000q_{60} = 20$, and $i = 6\%$
   Calculate $1000\ {}_{10}V_{50}$.

Solution:

1.

   (a) APV(FB) at time zero is given by

$$E\left[b_{T_x}e^{-\delta T_x}\right] = \int_0^\infty b_t e^{-\delta t}\ f_{65}(t)dt = 2000\int_0^\infty f_{65}(t)dt = 2000.$$

   APV(FP) at time zero is given by

$$\begin{aligned} P\bar{a}_{65} &= P\int_0^\infty e^{-\delta t}\ {}_tp_{65}dt = P\int_0^\infty e^{-0.05t}e^{-0.04t}dt \\ &= P\frac{100}{9}\int_0^\infty 0.09e^{-0.09t}dt = P\frac{100}{9}. \end{aligned}$$

   Therefore the annual net premium rate for this life insurance, given, by the equivalence principle is

$$P = \mathbf{180}$$

   (b) The APV, at time 2, of future benefit is

$$\begin{aligned} \mathbf{APV(FB)}_2 &= \int_0^\infty b_{t+2}e^{-\delta t}\ {}_tp_{67}\mu_{67+t}dt = \int_0^\infty 2000e^{0.05(t+2)}e^{-0.05t}\ e^{-0.04t}0.04dt \\ &= 2000\int_0^\infty e^{-0.02t}0.04dt = 2000e^{0.1}\int_0^\infty 0.04e^{-0.04t}dt = 2210.3418. \end{aligned}$$

   and the APV, at time 2, of future premium rates is $P\bar{a}_{67}$ where

$$\bar{a}_{67} = E\left[\frac{1-v^{T_{67}}}{\delta}\right] = \frac{1}{\mu+\delta} = \frac{1}{0.09} = \frac{100}{9}.$$

$$\begin{aligned} {}_2V &= \mathbf{APV(FB)}_2 - P\bar{a}_{67} = 2210.3418 - 180 \times \frac{100}{9} \\ &= 2210.3418 - 2000 = \mathbf{210.3418}. \end{aligned}$$

2. We know that

$$\begin{aligned} 1000\ {}_{10}V_{50} &= 1000\left(A_{60} - P_{50}\ddot{a}_{60}\right) = 1000A_{60} - 1000P_{50}\ddot{a}_{60} \\ &= 1000\ A_{60} - 25\ddot{a}_{60} = 1000A_{60} - 25\left(\frac{1 - A_{60}}{d}\right). \end{aligned}$$

Now, we need A_{60} By recursion relation for life insurance we can write

$$A_{60} = vq_{60} + vp_{60}\ A_{61},$$

then

$$\begin{aligned} 1000\ A_{60} &= v \times 1000\ q_{60} + vp_{60} \times 1000\ A_{61} \\ &= \frac{20}{1.06} + \frac{1 - 0.02}{1.06} \times 440 = 425.66 \end{aligned}$$

Consequently

$$1000\ {}_{10}V_{50} = 425.66 - 25\left(\frac{1 - 0.42566}{0.06}\right)(1.06) = \mathbf{171.99316}.$$

Problem 2. (6 marks)

For a fully discrete 20–year deferred whole life insurance of 1000 on (50), you are given: (i) Premiums are payable for 20 years. (ii) Deaths are Uniformly Distributed between integral ages. (iv) $\delta = \ln(1.045)$, $q_{59} = q_{60} = q_{70} = 0.5$ and ${}_9V = 60$, ${}_{10}V = 250$, ${}_{21}V = 1850$

1. **(2 marks)** Calculate the level net premium for this policy.
2. **(2 marks)** Calculate ${}_{11}V$, the net premium reserve at the end of year 11.
3. **(2 marks)** Calculate ${}_{20}V$, the net premium reserve at the end of year 20.

Solution:

1. Remark first that $i = 0.045$. From recursion formula we have

$$\begin{aligned} ({}_9V + P)(1+i) &= q_{59} \times 0 + {}_{10}V\,p_{59} \\ &= (1 - q_{59})\ {}_{10}V = \left(1 - \frac{1}{2}\right) 250 = 125.0 \end{aligned}$$

then

$$P = \frac{125.0}{1.045} - 60 = \mathbf{59.6172}$$

2. From recursion formula we have also

$$({}_{10}V + P)(1+i) = q_{60} \times 0 + {}_{11}V\,p_{60}$$

hence

$${}_{11}V = \frac{(250 + 59.6172) \times 1.045}{1 - \frac{1}{2}} = \mathbf{647.099948}$$

3. We have

$${}_{20}V\,(1.045) = q_{70} \times 1000 + {}_{21}V\,(1 - q_{70}) = \frac{1000}{2} + \frac{{}_{21}V}{2},$$

then

$${}_{20}V = \frac{1000 + 1850}{2\,(1.045)} = \mathbf{1363.636364}.$$

Problem 3. (6 marks)

1. **(2 marks)** For an insurance policy of $10,000$ on (70):
(i) Premiums are payable quarterly. The quarterly net premium is 175.
(ii) Benefits are paid at the end of the year of death.
(iii) $q_{78} = 0.04$, with constant force of mortality between integral ages.
(iv) $i = 0.1$ and ${}_{8.1}V = 1122$. Calculate ${}_{8.7}V$.

2. **(2 marks)** For a fully continuous 10–year endowment insurance of $100,000$ on (55):
(i) The annual gross premium is 250.
(ii) Expenses are 3% of the gross premium, plus settlement expenses of 100.
(iii) $\mu_x = 0.001(1.015)^x$ and $\delta = 0.04$
Using Euler's method **Lower end** with step $h = 0.1$ to calculate ${}_{9.8}V^g$.

3. **(2 marks)** For a whole life insurance of $20,000$ on (x), we assume that benefits are payable at the moment of death, level premiums are payable at the beginning of each year, Deaths are uniformly distributed over each year of age and $i = 0.04$, $\ddot{a}_x = 8$, $\ddot{a}_{x+10} = 6$,
Calculate the 10 year net premium reserve for this insurance.

Solution:

1. From recursion we have

$$\left({}_{8.1}V + 0\right)(1.1)^{0.25-0.1} = 10000\, v^{1-0.25}\; {}_{0.15}q_{78.1} + {}_{8.25}V\; {}_{0.15}p_{78.1}$$

$$\Updownarrow$$

$$\begin{aligned}
{}_{8.25}V &= \frac{\left({}_{8.1}V + 0\right)(1.1)^{0.15} - 10000 \times (1.1)^{-0.75}\,\left(1 - {}_{0.15}p_{78.1}\right)}{{}_{0.15}p_{78.1}} \\
&= \frac{\left({}_{8.1}V + 0\right)(1.1)^{0.15} - 10000 \times (1.1)^{-0.75}\left(1 - p_{78}^{0.15}\right)}{p_{78}^{0.15}} \\
&= \frac{(1122)(1.1)^{0.15} - 10000 \times (1.1)^{-0.75}\left(1 - (0.96)^{0.15}\right)}{(0.96)^{0.15}} = \mathbf{1087.9630}
\end{aligned}$$

and

$$\begin{aligned}
{}_{8.5}V &= \frac{\left({}_{8.25}V + 175\right)(1.1)^{0.25} - 10000 \times (1.1)^{-0.5}\left(1 - {}_{0.25}p_{78.25}\right)}{{}_{0.25}p_{78.25}} \\
&= \frac{\left({}_{8.25}V + 175\right)(1.1)^{0.25} - 10000 \times (1.1)^{-0.5}\left(1 - p_{78}^{0.25}\right)}{p_{78}^{0.25}} \text{ (since } 0.25 + 0.25 \leq 1) \\
&= \frac{(1087.9630 + 175)(1.1)^{0.25} - 10000 \times (1.1)^{-0.5}\left(1 - (0.96)^{0.25}\right)}{(0.96)^{0.25}} = \mathbf{1208.8814}
\end{aligned}$$

and

$$\begin{aligned}
{}_{8.7}V &= \frac{\left({}_{8.5}V + 175\right)(1.1)^{0.2} - 10000 \times (1.1)^{-0.3}\left(1 - {}_{0.2}p_{78.5}\right)}{{}_{0.2}p_{78.5}} \\
&= \frac{\left({}_{8.25}V + 175\right)(1.1)^{0.2} - 10000 \times (1.1)^{-0.3}\left(1 - p_{78}^{0.2}\right)}{p_{78}^{0.2}} \text{(since } 0.2 + 0.25 \leq 1) \\
&= \frac{(1208.8814 + 175)(1.1)^{0.2} - 10000 \times (1.1)^{-0.3}\left(1 - (0.96)^{0.2}\right)}{(0.96)^{0.2}} = \mathbf{1342.4098}.
\end{aligned}$$

2. We have $c_t = 0.03$, $e_t = 0$ and $\mu_x = 0.001(1.015)^x$ moreover

$$
\begin{aligned}
{}_tV^g &\simeq \frac{{}_{t+h}V^g - h\left(G_t - c_tG_t - (b_t+E_t)\,\mu_{x+t}\right)}{1+h\left(\delta_t+\mu_{x+t}\right)} \\
&= \frac{{}_{t+h}V^g - h\left(G_t\,(1-c_t) - (b_t+E_t)\,\mu_{x+t}\right)}{1+h\left(\delta_t+\mu_{x+t}\right)} \qquad (t=9.9, h=0.1)
\end{aligned}
$$

then

$$
{}_{9.9}V^g \simeq \frac{{}_{10}V^g - 0.1\left(250\,(1-0.03) - (100000+100)\,\mu_{55+9.9}\right)}{1+0.1\left(0.04+\mu_{55+9.9}\right)}
$$

but $\mu_{55+9.9} = \mu_{64.9} = 0.001(1.015)^{64.9} = 0.002628$ and ${}_{10}V^g = 0$ since this is a term insurance. Then

$$
{}_{9.9}V^g \simeq \frac{100000 - 0.1\,(250\,(1-0.03) - (100000+100)\,(0.002628))}{1+0.1\,(0.04+0.002628)} = \mathbf{99577.57698}.
$$

Again backward recursion and taking $(t = 9.8,\ h = 0.1)$ we get

$$
{}_{9.8}V^g \simeq \frac{{}_{9.9}V^g - 0.1\left(250\,(1-0.03) - (100000+100)\,\mu_{55+9.8}\right)}{1+0.1\left(0.04+\mu_{55+9.8}\right)}
$$

and $\mu_{55+9.8} = \mu_{64.8} = (0.001)\,(1.015)^{64.8} = 0.002624$, thus

$$
{}_{9.8}V^g \simeq \frac{99577.57698 - 0.1\,(250\,(1-0.03) - (100000+100)\,(0.002624))}{1+0.1\,(0.04+0.002624)} = \mathbf{99156.94665}.
$$

3. We know that for level benefits equals 1 we have ${}_{10}V = \bar{A}_{x+10} - P\ddot{a}_{x+10}$, where P the level net premium for a semi–annual whole life insurance of 1 on (x). We first calculate P. The APV of benefit is

$$
\bar{A}_x = \frac{i}{\delta}A_x = \frac{i}{\delta}(1-d\ddot{a}_x) = \frac{0.04}{\ln(1.04)}\left(1-\frac{0.04}{1.04}\,(8)\right) = 0.706063339
$$

From the equivalence principle we have $P = \frac{\bar{A}_x}{\ddot{a}_x} = \frac{0.706063339}{8} = 0.088257917$.
At the end of 10 years, the APV of future benefit is

$$
\bar{A}_{x+10} = \frac{i}{\delta}A_{x+10} = \frac{i}{\delta}(1-d\ddot{a}_{x+10}) = \frac{0.04}{\ln(1.04)}\left(1-\frac{0.04}{1.04}6\right) = 0.784514821
$$

Consequently, $20000\ {}_{10}V = 20000\,(0.784514821 - 0.088257917(6)) = \mathbf{5099.34638}$.

Problem 4. (6 marks)

1. **(2 marks)** For a fully discrete whole life insurance of 10,000 on (45):
(i) Both net and gross premiums are reduced by half starting with the 21^{st} premium.
(ii) Expenses are 50% of premium in the first year, 10% of premium in renewal years.
(iii) $A_{45} = 0.15$, $A_{65} = 0.24$, $A_{75} = 0.40$, $d = 0.04$, ${}_{20}p_{45} = 0.90$ and ${}_{30}p_{45} = 0.75$.
Calculate the expense reserve at time 30 given that the gross premium $G = 97.886$ and the net premium $P = 85.857$.

2. **(2 marks)** Calculate the FPT reserve at time 2 for a fully discrete 15–year endowment insurance of $1,000,000$ on (30) given that: $q_{31} = 0.002$, $\ddot{a}_{32:\overline{13}|} = 9$ and $i = 0.05$,

3. **(2 marks)** For a fully discrete whole life insurance of 1000 on (30), the modified premium for each of the first three years is 17.72, and modified premiums are level thereafter.
Calculate the modified reserve at the end of year 20 using Illustrative Life Table and $i = 0.06$

Solution:

1. The expense premium for the first 20 years is $P^e = 97.886 - 85.857 = 12.029 \simeq \mathbf{12.03}$.
   The expense reserve is the APV or EPV of future expenses minus expense premiums at time 30. Do not forget that both the gross and the expense premiums are reduced by half after 20 years.

$$\begin{aligned} {}_{30}V^e &= 0.1\frac{G}{2}\ddot{a}_{75} - \frac{P^e}{2}\ddot{a}_{75} = \frac{1}{2}\left(0.1G - P^e\right)\ddot{a}_{75} \\ &= \frac{1}{2}\left(0.1G - P^e\right)\left(\frac{1 - A_{75}}{d}\right) = 0.5\left(0.1 \times 97.89 - 12.03\right)\frac{1 - 0.4}{0.04} = -\mathbf{16.8075} \end{aligned}$$

2. We have

$$\begin{aligned} 10^6\ {}_2V^{\mathrm{FPT}} &= 10^6{}_1V_{31:\overline{14}|} = 10^5\left(1 - \frac{\ddot{a}_{32:\overline{13}|}}{\ddot{a}_{31:\overline{14}|}}\right) = 10^6\left(1 - \frac{\ddot{a}_{32:\overline{13}|}}{1 + vp_{31}\ddot{a}_{32:\overline{13}|}}\right) \\ &= 10^6\left(1 - \frac{9}{1 + \frac{1}{1.05}\left(1 - 0.002\right) \times 9}\right) = \mathbf{58014.35407}. \end{aligned}$$

3. We have $\mathrm{APV(FP)}_0 = \alpha\ddot{a}_{30:\overline{3}|} + \beta\ {}_{3|}\ddot{a}_{30} = P\ddot{a}_{30}$ where P is the level annual net premium which is given by

$$P = \frac{1000A_{30}}{\ddot{a}_{30}} = 1000\left(\frac{1}{\ddot{a}_{30}} - d\right) = 1000\left(\frac{1}{15.8561} - \frac{0.06}{1.06}\right) = 6.4634.$$

   Therefore

$$\begin{aligned} \beta &= \frac{P\ddot{a}_{30} - \alpha\ddot{a}_{30:\overline{3}|}}{{}_{3|}\ddot{a}_{30}} = \frac{P\ddot{a}_{30} - \alpha\left(\ddot{a}_{30} - {}_{3|}\ddot{a}_{30}\right)}{{}_{3|}\ddot{a}_{30}} \\ &= \frac{P\ddot{a}_{30} - \alpha\left(\ddot{a}_{30} - {}_3E_{30}\ddot{a}_{33}\right)}{{}_3E_{30}\ddot{a}_{33}} = \frac{P\ddot{a}_{30} - \alpha\left(\ddot{a}_{30} - v^3\frac{\ell_{33}}{\ell_{30}}\ddot{a}_{33}\right)}{v^3\frac{\ell_{33}}{\ell_{30}}\ddot{a}_{33}} \\ &= \frac{6.4634 \times 15.8561 - 17.72\left(15.8561 - \frac{1}{(1.06)^3}\frac{9455522}{9501381}15.5906\right)}{\frac{1}{(1.0)^3}\frac{9455522}{9501381}15.5906} = 4.0188. \end{aligned}$$

   The modified reserve is then

$$\begin{aligned} 1000\ {}_{20}V_{30}^{\mathrm{mod}} &= 1000A_{50} - \beta\ddot{a}_{50} = 1000A_{50} - \beta\left(\frac{1 - A_{50}}{d}\right) \\ &= 249.05 - 4.0188 \times \frac{1 - 0.24905}{0.056604} = \mathbf{195.733664}. \end{aligned}$$

Problem 5. (6 marks)

For a fully discrete 20–year term life insurance on (40), you are given:
(i) The death benefit is $10,000$.
(ii) The death benefit is payable at the end of the year of death.
(iii) Values in year 4:

	Anticipated	Actual
Gross annual premium	90	90
Expenses as a percent of premium	3%	2.5%
The death probability q_{43}	0.035	0.025
Annual effective rate of interest	5%	4%

(iii) Gross premium reserves are given as follows

End of year	Reserve
3	100
4	125

A company issued the 20–year term life insurance to 1000 lives age 40 with independent future lifetimes. At the end of the third year 990 insurances remain in force.

1. **(3 marks)** Calculate the total gain in year 4 from the 990 insurances.
2. **(3 marks)** Decompose the total gain in gains from **interest**, **expenses** and **mortality**.

Solution:

1. The **actual profit** is

$$\begin{aligned}\widehat{\mathrm{Pr}}_4 &= N\left[\left({}_3V^g + G_3\,(1-\widehat{c}_3) - \widehat{e}_3\right)\left(1+\widehat{i}_4\right)\right] - N\left[\left(b_4 + \widehat{E}_4\right)\widehat{q}_{43} + {}_4V^g\,\widehat{p}_{43}\right] \\ &= 990\,(100 + 90(1-0.025)) \times (1.04) - 990 \times (0.025 \times 10000 + 0.975 \times 125) = -174848.8500.\end{aligned}$$

Similarly, the **expected profit** is

$$\begin{aligned}\mathrm{Pr}_4 &= N\left[\left({}_3V^g + G_3\,(1-c_3) - e_3\right)(1+i_4)\right] - N\left[(b_4 + E_4)\,q_{43} + {}_4V^g\,p_{43}\right] \\ &= 990\,(100 + 90(1-0.03)) \times (1.05) - 990 \times (0.035 \times 10000 + 0.965 \times 125) = -271220.4000\end{aligned}$$

So the total gain is $-174848.8500 - (-271220.40) = \mathbf{96371.55}$.

2. The total $\mathbf{Gain}_4 = \mathrm{Pr}_4^i + \mathrm{Pr}_4^e + \mathrm{Pr}_4^d$ where
Interest component:

$$\begin{aligned}\mathrm{Pr}_4^i &= N\left[\left({}_3V^g + G_3\,(1-c_3) - e_3\right)\left(\widehat{i}_4 - i_4\right)\right] \\ &= 990\,(100 + 90(1-0.03))\,(0.04-0.05) = \mathbf{-1854.270}\end{aligned}$$

Expenses component:

$$\begin{aligned}\mathrm{Pr}_4^e &= -N\left(G_3\left(\widehat{c}_3 - c_3\right) + \widehat{e}_3 - e_3\right)\left(1+\widehat{i}_4\right) - N\left(\widehat{E}_4 - E_4\right) q_{43} \\ &= -990 \times 90 \times (0.025 - 0.03) \times 1.04 - 990 \times 0 \times 0.025 = \mathbf{463.320}\end{aligned}$$

Mortality component:

$$\begin{aligned}\mathrm{Pr}_4^d &= -N\left(b_4 + \widehat{E}_4 - {}_4V^g\right)\left(q_{43} - \widehat{q}_{43}\right) \\ &= -990 \times (10000 - 125) \times (0.025 - 0.035) = \mathbf{97762.500}.\end{aligned}$$

Verification $-1854.270 + 463.320 + 97762.500 = \mathbf{96371.55}$.