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Model Answer

Midterm Exam 1, Actuarial Mathematical Models 2

Exercise 1

1. *Individuals age (50) are subject to death by accident (decrement 1) or death by natural causes (decrement 2).*

$$\text{You are given } {}_tq_{50}^{(1)} = \frac{t}{60} \text{ and } {}_tq_{50}^{(2)} = \frac{t}{30} \text{ for } 0 < t \le 20$$

   Calculate $q_{58}^{(2)}$*, the probability that an individual age 58 will die of natural causes in the following year (Hint* ${}_{t|u}q_x^{(i)} = {}_tp_x^{(\tau)}\ {}_uq_{x+t}^{(i)} = {}_{t+u}q_x^{(i)} - {}_tq_x^{(i)}$*).*

2. *You are given the following triple–decrement table:*

k	$q_{x+k}^{(1)}$	$q_{x+k}^{(2)}$	$q_{x+k}^{(3)}$
0	0.10	0.01	0.02
1	0.05	0.02	0.03
2	0.01	0.03	0.02

   Calculate ${}_{2|}q_x^{(2)}$.

3. *For a double decrement model assume that :* $q_x^{(2)} = \frac{1}{8}$, ${}_{1|}q_x^{(1)} = \frac{11}{48}$, $q_{x+1}^{(1)} = \frac{1}{3}$ *Calculate* $q_x^{(1)}$.

Solution:

1. We have

$$\begin{aligned} q_{58}^{(2)} &= \frac{{}_{8|}q_{50}^{(2)}}{{}_8p_{50}^{(\tau)}} = \frac{{}_9q_{50}^{(2)} - {}_8q_{50}^{(2)}}{1 - {}_8q_{50}^{(\tau)}} = \frac{{}_9q_{50}^{(2)} - {}_8q_{50}^{(2)}}{1 - \left({}_8q_{50}^{(1)} + {}_8q_{50}^{(2)}\right)} \\ &= \frac{\frac{9}{30} - \frac{8}{30}}{1 - \left(\frac{8}{60} + \frac{8}{30}\right)} = \frac{1}{18} = \mathbf{0.055556} \end{aligned}$$

2. This is the probability of decrement due to cause (2) in year 3, or

$$\begin{aligned} {}_{2|}q_x^{(2)} &= {}_2p_x^{(\tau)}\ q_{x+2}^{(2)} = p_x^{(\tau)}\ p_{x+1}^{(\tau)}\ q_{x+2}^{(2)} \\ &= \left(1 - q_x^{(1)} - q_x^{(2)} - q_x^{(3)}\right)\left(1 - q_{x+1}^{(1)} - q_{x+1}^{(2)} - q_{x+1}^{(3)}\right)\left(q_{x+2}^{(2)}\right) \\ &= (1 - 0.10 - 0.01 - 0.02)(1 - 0.05 - 0.02 - 0.03)(0.03) = \mathbf{0.023490} \end{aligned}$$

3. We have

$$q_x^{(1)} = q_x^{(\tau)} - q_x^{(2)} = 1 - p_x^{(\tau)} - q_x^{(2)} = 1 - p_x^{(\tau)} - \frac{1}{8}$$

moreover

$$\begin{aligned} \frac{11}{48} &= {}_{1|}q_x^{(1)} = p_x^{(\tau)}\, q_{x+1}^{(1)} = p_x^{(\tau)}\left(\frac{1}{3}\right) \Longrightarrow p_x^{(\tau)} = \frac{11}{16} \\ q_x^{(1)} &= 1 - \frac{11}{16} - \frac{1}{8} = \frac{3}{16} = \mathbf{0.1875} \end{aligned}$$

Exercise 2

A 3–year term insurance on (40) pays 10,000 at the end of the year of death, and pays an additional 10,000 at the end of the year of death if death is due to accidental causes. You are given: (i) The following mortality table:

Age	*Probability of Death from Other Causes*	*Probability of Death from Accidental Causes*
40	0.015	0.002
41	0.016	0.002
42	0.017	0.002

(ii) $i = 0.05$ *Calculate the net single premium for the insurance.*

Solution: Let us calculate the needed probabilities.

$$\begin{aligned} p_{40}^{(\tau)} &= 1 - 0.015 - 0.002 = 0.983 \\ {}_2p_{40}^{(\tau)} &= (0.983)(1 - 0.016 - 0.002) = 0.965306 \end{aligned}$$

The APV of the insurance is given

$$\begin{aligned} \mathbf{NSP} &= \frac{(0.015(10000) + 0.002(20000))}{1.05} + \frac{0.983(0.016(10000) + 0.002(20000))}{1.05^2} \\ &+ \frac{0.965306(0.017(10000) + 0.002(20000))}{1.05^3} \\ &= \mathbf{534.39} \end{aligned}$$

Exercise 3

A 3–year term life insurance policy pays 10000 at the end of the year of death. Premiums are paid at the beginning of each year. However, if the insured is disabled at the beginning of the year, the premium is waived. The ***Active state*** *denotes a non–disabled insured, and the* ***Disabled state*** *a disabled insured. The probability of an insured being in a state at the beginning of the year, given the state at the beginning of the previous year, is as follows: beginning of the*

State	***State next year***		
previous year	***Active***	***Disabled***	***Dead***
Active	0.80	0.19	0.01
Disabled	0.20	0.78	0.02
Dead	0	0	1

For an individual currently in the Active state, calculate the total annual net premium for the death benefits of the life insurance policy, taking into account that premiums are only paid in the Active state and $i = 0.06$.

<u>**Solution**</u>:

The state probability vector at the end of the first year is $(0.8,\ 0.19,\ 0.01)$. The state probability vector at the end of the second year is

$$\begin{pmatrix} 0.8 & 0.19 & 0.01 \end{pmatrix} \begin{pmatrix} 0.8 & 0.19 & 0.01 \\ 0.20 & 0.78 & 0.02 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.678 & 0.3002 & 0.0218 \end{pmatrix}$$

The probability of dying is 0.01 in the first year, $0.01(0.8) + 0.02(0.19) = 0.0118$ in the second year, and $0.01(0.678) + 0.02(0.3002) = 0.012784$ in the third year. The actuarial present value of the death benefit is

$$10000 \left(\frac{0.01}{1.06} + \frac{0.0118}{1.06^2} + \frac{0.012784}{1.06^3} \right) = 306.696$$

The expected present value of a premium of 1 per year is

$$1 + \frac{0.8}{1.06} + \frac{0.678}{1.06^2} = 2.35813$$

The net premium is $P = \frac{306.696}{2.35813} = \mathbf{130.059}$.

Exercise 4

You are given the following Markov chain as a model for disability insurance: You are also given: (i) $\mu^{01}_{x+t} = 0.01,\ t > 0$ *(ii)* $\mu^{02}_{x+t} = 0.02,\ t > 0$ *(iii)* $\mu^{03}_{x+t} = 0.10,\ t > 0$ *(iv)* $\mu^{12}_{x+t} = 0.05,\ t > 0$ *(v)* $\delta = 0.04$*. A* 10*–year disability insurance on* (x) *pays* 1000 *per year continuously while an insured is disabled. Premiums are payable continuously for* 10 *years while healthy and are determined using the equivalence principle.*
Calculate the annual premium. (Hint ${}_tp^{01}_x = 0.125\left(e^{-0.05t} - e^{-0.13t}\right)$*).*

<u>**Solution**</u>:

The APV of 1 unit of premium annuity for 10 years, by the usual formula for an annuity with total decrement $0.01 + 0.02 + 0.10 = 0.13$ and $\delta = 0.04$, is

$$\bar{a}^{00}_{x:\overline{10}|} = \frac{1 - e^{-0.17(10)}}{0.13 + 0.04} = 4.80774.$$

For the disability annuity, we need to calculate $\int_0^{10} v^t\ {}_tp^{01}_x\ \mathrm{d}t$.

$${}_tp^{01}_x = 0.01 \int_0^t e^{-0.13u} e^{-0.05(t-u)} \mathrm{d}u = 0.01 e^{-0.05t} \frac{1 - e^{-0.08t}}{0.08} = 0.125\left(e^{-0.05t} - e^{-0.13t}\right)$$

$$\begin{aligned} \text{APV(disability annuity)} &= 1000\bar{a}^{01}_{x:\overline{10}|} = 1000(0.125) \int_0^{10} e^{-0.04t}\left(e^{-0.05t} - e^{-0.13t}\right) \mathrm{d}t \\ &= 1000(0.125) \int_0^{10} \left(e^{-0.09t} - e^{-0.17t}\right) \mathrm{d}t \\ &= 125 \left(\frac{1 - e^{-0.9}}{0.09} - \frac{1 - e^{-1.7}}{0.17} \right) = 223.241 \end{aligned}$$

The annual premium is

$$P = \frac{1000\bar{a}^{01}_{x:\overline{10}|}}{\bar{a}^{00}_{x:\overline{10}|}} = \frac{223.241}{4.80774} = \mathbf{46.4336}.$$

Exercise 5

A 20–year disability income policy on (x) is modeled with the following Markov chain: You are given: (i) Transition forces $\mu^{01}_{x+10} = 0.10$, $\mu^{02}_{x+10} = 0.04$, $\mu^{10}_{x+10} = 0.08$, $\mu^{12}_{x+10} = 0.05$. (ii) $\mu^{ij}_{x+9.5} = \mu^{ij}_{x+10}$ for all i and j. (iii) $\delta = 0.08$ (iv) The policy pays a continuous annuity benefit of 1200 per year while sick and a death benefit of 20000 at the moment of death. (v) Continuous premiums for the policy are 200 per year, paid only when healthy. (vi) ${}_{10}V^{(0)} = 2250$, ${}_{10}V^{(1)} = 11000$.
Calculate ${}_{9}V^{(1)}$ using Euler's method with step 0.5 to numerically solve Thiele's differential equation. Notice that while in state (2) at time t we have ${}_{t}V^{(2)} = 0$.

Hint

$$\bar{a}^{ij}_x = \int_0^\infty e^{-\delta t}\,{}_tp^{ij}_x\,\mathrm{d}t, \quad \ddot{a}^{ij}_x = \sum_{k=0}^{\infty} v^k\,{}_kp^{ij}_x, \quad \bar{A}^{ij}_x = \int_0^\infty \sum_{k\neq j} e^{-\delta t}\,{}_tp^{ik}_x\,\mu^{kj}_{x+t}\,\mathrm{d}t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}_tV^{(i)}\right) = \delta_t\,{}_tV^{(i)} - \left(S^{(i)}_t - P^{(i)}_t\right) - \sum_{j=0,\ j\neq i}^{n} \mu^{ij}_{x+t}\left(b^{(ij)}_t + {}_tV^{(j)} - {}_tV^{(i)}\right)$$

(Euler's method for Thiele's differential equation)

$${}_{t-h}V^{(i)} \approx {}_tV^{(i)}\,(1-\delta_t h) + h\left(S^{(i)}_t - P^{(i)}_t\right) + h\sum_{j=0,\ j\neq i}^{n} \mu^{ij}_{x+t}\left(b^{(ij)}_t + {}_tV^{(j)} - {}_tV^{(i)}\right)$$

<u>**Solution**</u>:

$$\begin{aligned}
{}_{9.5}V^{(0)} &= {}_{10}V^{(0)}\,(1-\delta h) + h\left(S^{(0)} - P^{(0)}\right) + h\sum_{j=1}^{2} \mu^{0j}_{x+10}\left(b^{(0j)}_{10} + {}_{10}V^{(j)} - {}_{10}V^{(0)}\right)\\
&= 2250(1-0.5(0.08)) + 0.5(0-200) + 0.5(0.10(0+11000-2250) + 0.04(20000+0-2250))\\
&= \mathbf{2852.5}\\
{}_{9.5}V^{(1)} &= {}_{10}V^{(1)}\,(1-\delta h) + h\left(S^{(1)} - P^{(1)}\right) + h\sum_{j=0,2} \mu^{1j}_{x+10}\left(b^{(1j)}_{10} + {}_{10}V^{(j)} - {}_{10}V^{(1)}\right)\\
&= 11000(1-0.5(0.08)) + 0.5(1200-0) + 0.5(0.08(0+2250-11000) + 0.05(20000+0-11000))\\
&= \mathbf{11035}\\
{}_{9}V^{(1)} &= {}_{9.5}V^{(1)}\,(1-\delta h) + h\left(S^{(1)} - P^{(1)}\right) + h\sum_{j=0,2} \mu^{1j}_{x+9.5}\left(b^{(1j)}_{9.5} + {}_{9.5}V^{(j)} - {}_{9.5}V^{(1)}\right)\\
&= 11035(1-0.5(0.08)) + 0.5(1200-0) + 0.5(0.08(0+2852.5-11035) + 0.05(20000+0-11035))\\
&= \mathbf{11090.43}
\end{aligned}$$