King Saud University
College of Sciences
Mathematics Department

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Model Answer of the second midterm exam ACTU–462 (25%)

December 2, 2019 (Fall 2019)

<u>Problem 1.</u> (6 marks)

1. **(2 mark)** For a special fully discrete 3–year term life insurance of 25,000 on (50), you are given: (i) The annual effective rate of interest is 4%. (ii) The net premium in year 1 is $25000A^{1}_{50:\overline{1}|}$. (iii) The net premium in years 2 and 3 are equal. (iv) The mortality table has the following values:

x	50	51	52	53
q_x	0.05	0.06	0.07	0.08

Calculate the net premium reserve at the end of year 2.

2. **(2 marks)** For a fully discrete, level premium, 5–year term insurance on (40) you are given: (i) Mortality follows the Illustrative Life Table and $i = 0.06$. Death benefit is 5000. (ii) Expenses, paid at the beginning of the policy year, are given as follows:

	Per policy	Fraction of premium
in the First year	120	0.7
in renewal years	25	0.1

Calculate the gross level annual premium using equivalence principle.

3. **(2 marks)** For a fully discrete 20–year term insurance of 100,000 on (35), you are given: (i) Annual gross premiums are 200. (ii) $q_{53} = 0.006$ and $q_{54} = 0.007$ and $i = 0.05$. (iii) annual expenses are 5% of premium plus 25, paid at the beginning of the year. (iv) Settlement expenses are 100. Calculate the gross premium reserve at the end of year 18.

Solution:

1. The reserve at 2 is given by ${}_2V = \text{APV}(\text{FB})_1 - \text{APV}(\text{FP})_1 = 25000\ vq_{52} - P$. We have first to find P. Since the net premium in year 1 equals the APV of the year–1 death benefit, the first year can be ignored in the calculation of the premium. In fact, this is a full preliminary term reserve. So we can calculate the premium starting at the end of 1 year, looking prospectively at the next 2 years

$$A^{1}_{51:\overline{2}|} = \frac{0.06}{1.04} + \frac{0.94 \times 0.07}{(1.04)^2} = 0.11853$$

and $\ddot{a}_{51:\overline{2}|} = 1 + \frac{0.94}{1.04} = 1.9038$. therefore

$$P = 25000\frac{0.11853}{1.9038} = 1556.5.$$

Therefore

$${}_2V = 25000\frac{0.07}{1.04} - 1556.5 = \mathbf{126.19}.$$

2. Let us first calculate $A^{\ 1}_{40:\overline{5}|}$ and $\ddot{a}_{40:\overline{5}|}$. We have

$$\begin{aligned} A^{\ 1}_{40:\overline{5}|} &= A_{40} - {}_5E_{40} A_{45} = 0.16132 - 0.73529\,(0.20120) = 0.013380 \\ \ddot{a}_{40:\overline{5}|} &= \ddot{a}_{40} - {}_5E_{40} \ddot{a}_{45} = 14.8166 - 0.73529\,(14.1121) = 4.4401 \end{aligned}$$

By the equivalence principle, the gross premium is given by solving the following equation

$$\begin{aligned} G\ddot{a}_{40:\overline{5}|} &= 5000 A^{\ 1}_{40:\overline{5}|} + 120 + 0.7G + (25 + 0.1G)\, a_{40:\overline{4}|} \\ 4.4401G &= 5000 \times 0.013380 + 95 + 0.6G + (25 + 0.1G)\, 4.4401 \end{aligned}$$

Solving for G we get $G = \mathbf{80.358}$

3. We will use two recursions. Annual expenses are $0.05(200) + 25 = 35$, so $G - e = 200 - 35 = 165$. The total payment on death including settlement expenses is $100,000 + 100 = 100,100$.

$$\left({}_{19}V + G - e - 0.05G\right)(1 + i) = q_{54}\,(100000 + 100) + {}_{20}V\ p_{54}$$

$${}_{19}V = \frac{q_{54}\,(100100) + {}_{20}V\,(1 - q_{54})}{1 + i} + e + 0.05G - G$$

since ${}_{20}V = 0$ and $25 + 0.05 \times 200 - 200 = -165$, we obtain

$${}_{19}V = \frac{0.007\,(100100)}{1.05} - 165 = 502.3333.$$

Use recursion between 18 and 19 we get

$$\begin{aligned} {}_{18}V &= \frac{q_{53}\,(100100) + {}_{19}V\,(1 - q_{53})}{1 + i} + e + 0.05G - G \\ &= \frac{0.006\,(100100) + 502.3333 \times (1 - 0.006)}{1.05} - 165 = \mathbf{882.5422} \end{aligned}$$

Remark that one can do it prospectively.

<u>**Problem 2.**</u> **(6 marks)**

1. **(3 mark)** For a fully discrete whole life insurance of 1 on (x), you are given: (i) Benefits are paid at the end of the year of death. (ii) $a_x = 4.8961$ and $a_{x+1} = 4.6448$ (iii) $i = 0.06$. Calculate the net premium for the first year under the full preliminary term method.

2. **(3 marks)** For a fully discrete 20–year endowment insurance of 1 on (45). You are given $i = 0.06$, $1000q_{46} = 4.24$ and $\ddot{a}_{47:\overline{18}|} = 10.918$. Calculate the FPT reserve at the end of year 2.

Solution:

1. We use annuity recursion to back out p_x.

$$\begin{aligned} \ddot{a}_x &= 1 + vp_x \ddot{a}_{x+1} \Longleftrightarrow p_x = \frac{\ddot{a}_x - 1}{v\ddot{a}_{x+1}} \\ \text{hence } p_x &= (1 + i)\,\frac{a_x}{1 + a_{x+1}} = 1.06\frac{4.8961}{5.6448} = 0.91941, \end{aligned}$$

then $q_x = 1 - 0.91941 = 0.08059$. Consequently the first year net premium under full preliminary term is the present value of a one–year term policy on (x), or $vq_x = \frac{q_x}{1+i} = \frac{0.08059}{1.06} = \mathbf{0.076028}$.

2. The FPT reserve at time 2 is the time 1 level net premium reserve for a 19–year endowment insurance on (46), so we have
$$ {}_2V^{\text{FPT}} = 1 - \frac{\ddot{a}_{47:\overline{18}|}}{\ddot{a}_{46:\overline{19}|}}. $$
Now, we need $\ddot{a}_{46:\overline{19}|}$, which backed up by recursion on annuities.
$$ \ddot{a}_{46:\overline{19}|} = 1 + vp_{46}\ddot{a}_{47:\overline{18}|} = 1 + \frac{1-0.00424}{1.06}10.918 = 11.25633 $$
Then net premium reserves is ${}_2V^{\text{FPT}} = 1 - \frac{10.918}{11.25633} = \mathbf{0.030057}$.

Problem 3. (6 marks)

1. **(3 marks)** For a fully discrete whole life insurance of 50000 on (55), the valuation premium in the first year is the same as under the full preliminary term method, the valuation premium in years 21 and later is the level net annual premium for the policy and the valuation premium in years $2-20$ is level. Given $i = 6\%$, $q_{55} = 0.00901$, ${}_{20}q_{55} = 0.30265$, $a_{55} = 13.3561$, $a_{75} = 8.6749$ and $\ddot{a}_{80} = 5.9050$. Calculate the valuation premium in years $2-20$.

2. **(3 marks)** Calculate ${}_{25}V^{\text{mod}}$, the modified reserve at the end of year 25.

Solution:

1. Let α be the first year valuation premium and β the valuation premium in years $2-20$. P will be the level net premium. The equivalence principle gives
$$ 50000A_{55} = \alpha + \beta\ {}_{1|}\ddot{a}_{55:\overline{19}|} + P\ {}_{20|}\ddot{a}_{55}. $$
So $\ddot{a}_{40:\overline{5}|} = \ddot{a}_{40} - {}_5E_{40}\ddot{a}_{45}$
$$ \beta = \frac{50000A_{55} - \alpha - P\ {}_{20}E_{55}\ddot{a}_{75}}{\ddot{a}_{55:\overline{20}|}} - 1 = \frac{50000\left(1 - d\ddot{a}_{55}\right) - \alpha - P\ {}_{20}E_{55}\ddot{a}_{75}}{\ddot{a}_{55} - {}_{20}E_{55}\ddot{a}_{75} - 1}. $$
Let us now find α and P. We know $\alpha = 50000vq_{55} = 50000\frac{0.00901}{1.06} = 425$, and
$$ P = 50000\left(\frac{1}{\ddot{a}_{55}} - d\right) = 50000\left(\frac{1}{1+a_{55}} - d\right) = 50000\left(\frac{1}{14.3561} - \frac{0.06}{1.06}\right) = 652.65. $$
Therefore
$$ \beta = \frac{50000\left(1 - \frac{0.06}{1.06}14.3561\right) - 425 - 652.65\left(1.06\right)^{-20}\left(1-0.30265\right)\left(9.6749\right)}{14.3561 - \left(1.06\right)^{-20}\left(1-0.30265\right)\left(9.6749\right) - 1} = \mathbf{672.88} $$

2. Now,
$$ \begin{aligned} {}_{25}V^{\text{mod}} &= 50000A_{80} - P\ddot{a}_{80} = 50000\left(1 - d\ddot{a}_{80}\right) - P\ddot{a}_{80} \\ &= 50000 - \left(P + d\right)\ddot{a}_{80} = 50000 - \left(652.65 + \frac{0.06}{1.06}\right)(5.9050) = \mathbf{29434}. \end{aligned} $$

Problem 4. (6 marks)

In an employee group, employees leave due to death (cause 1), termination (cause 2), or retirement (cause 3). The probabilities of each of these decrements is in the following table

Age	Death Probability	Withdrawal Probability	Retirement Age Probability
x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
55	0.01	0.07	0.02
56	0.01	0.06	0.03
57	0.02	0.05	0.04
58	0.02	0.04	0.05
59	0.03	0.04	0.07

1. **(3 marks)** Calculate the probability of an employee age 55 retiring at age 58 or 59.
2. **(3 marks)** Construct a life table for five years given $\ell_{55}^{(\tau)} = 100,000$. (Hint matrix with the first column is $x = 55, \ldots, 59$ and the first line $\ell_x^{(\tau)}$, $d_x^{(1)}$, $d_x^{(2)}$, $d_x^{(3)}$, $d_x^{(\tau)}$)

Solution:

1. We calculate $p_x^{(\tau)}$ for each age.

x	$p_x^{(\tau)} = 1 - q_x^{(\tau)} = 1 - q_x^{(1)} - q_x^{(2)} - q_x^{(3)}$
55	$1 - 0.01 - 0.07 - 0.02 = 0.9$
56	$1 - 0.01 - 0.06 - 0.03 = 0.9$
57	$1 - 0.02 - 0.05 - 0.04 = 0.89$
58	$1 - 0.02 - 0.04 - 0.05 = 0.89$

Then the probability of retiring at age 58 is

$$_{3|}q_{55}^{(3)} = {}_3p_{55}^{(\tau)}\ q_{58}^{(3)} = (0.9)(0.9)(0.89)(0.05) = 0.036045$$

and the probability of retiring at age 59 is

$$_{4|}q_{55}^{(3)} = {}_4p_{55}^{(\tau)} q_{59}^{(3)} = (0.9)(0.9)(0.89)(0.89)(0.07) = 0.044912,$$

hence the probability of retiring at 58 or 59 is $0.036045 + 0.044912 = \mathbf{0.080957}$.

2. To construct the life table we shall use the following formulas

$$d_x^{(i)} = q_x^{(i)} \ell_x^{(\tau)}, d_x^{(\tau)} = \sum_{i=1}^{3} d_x^{(i)} \quad \text{and} \quad \ell_{x+1}^{(\tau)} = \ell_x^{(\tau)} - d_x^{(\tau)}$$

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$	$d_x^{(\tau)}$
55	**100000**	**1000**	**7000**	**2000**	**10000**
56	**90000**	**900**	**5400**	**2700**	**9000**
57	**81000**	**1620**	**4050**	**3240**	**8910**
58	**72090**	**1442**	**2884**	**3605**	**7930**
59	**64160**	**1925**	**2566**	**4491**	**8982**

Problem 5. (6 marks)

1. **(3 marks)** A fully continuous whole life insurance contract on (60) pays a 2 benefit at the moment of death, plus an additional 3 at the moment of accidental death if the accidental death occurs before age 75. You are given: (i) The force of accidental death is a constant 0.005. (ii) Therefore of non–accidental death is a constant 0.02. (iii) $\delta = 0.05$. Calculate the annual net premium rate.

2. **(3 marks)** Consider a double decrement table with two causes of decrement: (1) death and (2) withdrawal

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
40	10000	500	5
41	9495	491	4
42	9000	507	3
43	8490	300	7

A 3–year **TERM** insurance is issued to a life aged (40). The benefit, payable at the end of the year, is 20000. If the life withdraws in the second, a withdrawal benefit of 30% of all the **premiums paid is returned**. There is an initial expense of 50, and commission is 5% of each premium. The annual effective interest rate is 6%. Calculate the expense–loaded reserve ${}_2V^g$ and ${}_3V^g$.

<u>**Solution**</u>:

1. This insurance can be seen as a whole life of 2 on (60) whatever the cause of death plus 15–year term insurance of 3 on (60) if death occurs by accident, so

$$\begin{aligned}\text{APV(FB)}_0 &= 2\int_0^\infty e^{-\delta t}\ {}_tp_x^{(\tau)}\mu^{(\tau)}dt + 3\int_0^{15} e^{-\delta t}\ {}_tp_x^{(\tau)}\mu^{(non-acc)}dt \\ &= 2\int_0^\infty e^{-0.075t}0.025dt + 3\int_0^{15} e^{-0.075t}0.005dt \\ &= 0.666\,67 + 0.135\,07 = 0.80174.\end{aligned}$$

$\text{APV(FP)}_0 = P\bar{a}_{60}$ and

$$\begin{aligned}\bar{a}_{60} &= \int_0^\infty e^{-\delta t}\ {}_tp_x^{(\tau)}dt = \int_0^\infty e^{-\delta t}e^{-\mu^{(\tau)}t}dt = \frac{1}{\delta+\mu^{(\tau)}} = \frac{1}{0.05+0.005+0.02} \\ &= \frac{1}{0.075} = \frac{1000}{75} = \frac{40}{3} = 13.333\end{aligned}$$

The annual net premium rate is $P = \frac{0.80174}{13.3333} = \mathbf{0.060131}$.

2. First we have to find the gross premium G using E.P.

$$\text{APV(FP)}_0 = G\ddot{a}_{40:\overline{3}|} = G\left(1 + \frac{1}{1.06}\frac{9495}{10000} + \frac{1}{1.06^2}\frac{9000}{10000}\right) = 2.6968G.$$

$$\begin{aligned}\text{APV(FB)}_0^{(1)} &= 20000\left(\sum_{k=0}^{2} v^{k+1}\ {}_{k|}q_{40}^{(1)}\right) = 20000\left(\sum_{k=0}^{2} v^{k+1}\frac{d_{40+k}^{(1)}}{\ell_{40}^{(\tau)}}\right) \\ &= \frac{20000}{\ell_{40}^{(\tau)}}\left(\sum_{k=0}^{2} v^{k+1}d_{40+k}^{(1)}\right) = \frac{20000}{\ell_{40}^{(\tau)}}\left(\sum_{k=0}^{2} v^{k+1}d_{40+k}^{(1)}\right) \\ &= 2\left(\frac{d_{40}^{(1)}}{1.06} + \frac{d_{41}^{(1)}}{1.06^2} + \frac{d_{42}^{(1)}}{1.06^3}\right) \\ &= 2\left(\frac{500}{1.06} + \frac{491}{1.06^2} + \frac{507}{1.06^3}\right) = 2668.7\end{aligned}$$

$$\begin{aligned}
\text{APV(FB)}_0^{(2)} &= \sum_{k=0}^{2} b_{k+1}^{(2)} v^{k+1} \ {}_{k|}q_{40}^{(2)} = b_2^{(2)} v^2 \ {}_{1|}q_{40}^{(2)} \\
&= 0.3 \left(\frac{2G}{1.06} \frac{d_{41}^{(2)}}{\ell_{40}^{(\tau)}} \right) = \frac{0.3G}{\ell_{40}^{(\tau)}} \left(\frac{2}{1.06} d_{41}^{(2)} \right) \\
&= \frac{0.3G}{10000} \left(\frac{2}{1.06} 4 \right) = \frac{0.3G}{10000} \left(\frac{8}{1.06} \right) = 0.00022642G.
\end{aligned}$$

$\text{APV(all future benefits+all expenses)}_0 = \text{APV(all future premiums)}_0$ that is

$$2.6968G = 50 + 2668.7 + 0.00022642G + 0.05G\,(2.6968)\,, \text{ hence } G = \mathbf{1061.3}.$$

It is clear that ${}_3V^g = 0$ since it is a 3–year term insurance. Now, we use recursion to back–up find ${}_2V^g$, we have and

$$\left({}_2V^g + G - 0.05G \right)(1+i) = 20000 q_{42}^{(1)} + 0.3\,(2)\,(1061.3)\, q_{42}^{(2)} + \ {}_3V^g \ p_{42}^{(\tau)}$$

$$\begin{aligned}
{}_2V^g &= \frac{20000 q_{42}^{(1)} + 0.3\,(2)\,(1061.3)\, q_{42}^{(2)}}{1+i} - 0.95G \\
&= \frac{20000 \frac{507}{9000} + 0.3\,(2)\,(1061.3)\, \frac{3}{9000}}{1.06} - 0.95 \times 1061.3 = \mathbf{17860}.
\end{aligned}$$