King Saud University
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Project ACTU. 464 Spring 2020 (10 Marks)

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Group 1.

A fire insurance company covers 200 contracts against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Class k	1	2	3	4	5
Contract amount in thousand	2	3	4	5	6
Number of contracts	80	40	20	30	30
Probability of occurrence of the claim	0.01	0.02	0.03	0.04	0.05

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to the **contract amount** and the parameter of the exponential density is **1**. Let S be the amount aggregate claims in a 1–year period.

1. Calculate the mean and variance of S.

2. Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the $100\alpha^{th}$ percentile of the distribution of total claims for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$.

Group 2.

A fire insurance company covers 200 contracts against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Class k	1	2	3	4	5
Contract amount in thousand	2	3	4	5	6
Number of contracts	80	40	20	30	30
Probability of occurrence of the claim	0.2	0.2	0.2	0.2	0.2

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to the **double** of the **contract amount** and the parameter of the exponential density is **2**. Let S be the amount aggregate claims in a 1–year period.

1. Calculate the mean and variance of S.

2. Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the $100\alpha^{th}$ percentile of the distribution of total claims for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$.

Group 3.

A fire insurance company covers 200 contracts against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Class k	1	2	3	4	5
Contract amount in thousand	2	3	4	5	6
Number of contracts	80	40	20	30	30
Probability of occurrence of the claim	0.2	0.2	0.2	0.2	0.2

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to **three times** of the **contract amount** and the parameter of the exponential density is **3**. Let S be the amount aggregate claims in a 1–year period.

1. Calculate the mean and variance of S.

2. Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the $100\alpha^{th}$ percentile of the distribution of total claims for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$.

Group 4.

Let X_i, for $i = 1, 2, 3$ be independent and identically distributed with the common c.d.f.

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad x \ge 1. \end{cases}$$

1. Use convolution and excel to fund the c.d.f. of $S = X_1 + X_2 + X_3$,

2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70, 0.75,\ 0.80,\ 0.85, 0.90, 0.95$ corresponding to the the aggregate loss S.

Solution:

Calculation of the p.d.f. It is clear that $S = X_1 + X_2 + X_3$ takes it values in the interval $[0, 3]$,

$$\begin{aligned} f_S(s) &= \frac{1}{2}\sum_{k=0}^{[s]} \frac{6}{k!(3-k)!}(-1)^k (s-k)^2 \\ &= \begin{cases} \frac{1}{2}s^2 & \text{if} \quad 0 < s < 1 \\ \frac{1}{2}s^2 - \frac{3}{2}(s-1)^2 & \text{if} \quad 1 \le s < 2 \\ \frac{1}{2}s^2 - \frac{3}{2}(s-1)^2 + \frac{3}{2}(s-2)^2 & \text{if} \quad 2 \le s < 3 \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \end{aligned}$$

Calculation of the c.d.f.

$$F_S(s) = \begin{cases} 0 & \text{if} \quad s<0 \\ \int_0^s \frac{1}{2}t^2 dt & \text{if} \ \ 0 \le s < 1 \\ \int_0^s \left(\frac{1}{2}t^2 - \frac{3}{2}(t-1)^2\right) dt & \text{if} \ \ 1 \le s < 2 \\ \int_0^s \left(\frac{1}{2}t^2 - \frac{3}{2}(t-1)^2 + \frac{3}{2}(t-2)^2\right) dt & \text{if} \ \ 2 \le s < 3 \\ 1 & \text{if} \quad s \ge 3 \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad s<0 \\ \frac{1}{6}t^3 & \text{if} \ \ 0 \le s < 1 \\ F_S(1) + \int_1^s \left(\frac{1}{2}t^2 - \frac{3}{2}(t-1)^2\right) dt & \text{if} \ \ 1 \le s < 2 \\ F_S(2) + \int_2^s \left(\frac{1}{2}t^2 - \frac{3}{2}(t-1)^2 + \frac{3}{2}(t-2)^2\right) dt & \text{if} \ \ 2 \le s < 3 \\ 1 & \text{if} \quad s \ge 3 \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad s<0 \\ \frac{1}{6}t^3 & \text{if} \ \ 0 \le s < 1 \\ \frac{1}{6} - \frac{1}{3}s^3 + \frac{3}{2}s^2 - \frac{3}{2}s + \frac{1}{3} = \frac{1}{6}t^3 - \frac{1}{2}\left(t-1\right)^3 & \text{if} \ \ 1 \le s < 2 \\ \frac{5}{6} + \frac{1}{6}s^3 - \frac{3}{2}s^2 + \frac{9}{2}s - \frac{13}{3} = 1 - \frac{1}{6}(3-t)^3 & \text{if} \ \ 2 \le s < 3 \\ 1 & \text{if} \quad s \ge 3 \end{cases}$$

Observe that

$$\frac{1}{6} + \int_1^s \left(\frac{1}{2}t^2 - \frac{3}{2}(t-1)^2\right) dt = \frac{1}{6}t^3 - \frac{1}{2}\left(t-1\right)^3$$

Observe that $F_S(1) = \frac{1}{6} = 0.16667$ and $F_S(2) = 0.83333$, therefore the solutions of the equations

$$F_S(s) = 0.70, \text{ or } 0.75, \text{ or } 0.80$$

belong to the interval $(1;2)$. Hence
the equation $\frac{1}{6}t^3 - \frac{1}{2}\left(t-1\right)^3$, gives $P_{0.70} = 1.7760$,
the equation $\frac{1}{6}t^3 - \frac{1}{2}\left(t-1\right)^3 = 0.75$, gives $P_{0.75} = 1.8529$.
the equation $\frac{1}{6}t^3 - \frac{1}{2}\left(t-1\right)^3 = 0.80$, gives $P_{0.80} = 1.9371$.

And the solutions of the equations

$$F_S(s) = 0.85, \text{ or } 0.90, \text{ or } 0.95$$

belong to the interval $(2;3)$. Hence
the equation $1 - \frac{1}{6}(3-s)^3 = 0.85$, gives $P_{0.85} = 2.0345$,
the equation $1 - \frac{1}{6}(3-s)^3 = 0.90$, gives $P_{0.90} = 2.1566$,
the equation $1 - \frac{1}{6}(3-s)^3 = 0.95$, gives $P_{0.85} = 2.3306$.

Group 5.

Let f_1, f_2 and f_3 be given p.m.f. corresponding the random variables X_1, X_2 and X_3 as follows

x	0	1	2	3	4
$f_1(x)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$		
$f_2(x)$	$\frac{1}{5}$		$\frac{4}{5}$		
$f_3(x)$	$\frac{1}{5}$		$\frac{3}{5}$		$\frac{1}{5}$

1. Use convolution and excel to fund the c.d.f. of $S = X_1 + X_2 + X_3$,
2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85, 0.90, 0.95$ corresponding to the the aggregate loss S.

Solution:
The set of all possible values of $S = X_1 + X_2$ is $\{0, 1, 2, 3, 4\}$ so for all $s \in \{0, 1, 2, 3, 4\}$,

$$f_{1+2}(s) = \sum_{x=0,2} f_{X_1}(s-x) f_{X_2}(x) = f_{X_1}(s) f_{X_2}(0) + f_{X_1}(s-2) f_{X_2}(2),$$

The set of all possible values of $S = X_1 + X_2 + X_3$ is $\{0, 1, 2, 3, 4, 5, 6, 4, 8\}$ so for all $s \in \{0, 1, 2, 3, 4, 5, 6, 4, 8\}$,

$$f_{1+2+3}(s) = \sum_{x=0,2,4} f_{1+2}(s-x) f_{X_3}(x) = f_{1+2}(s) f_{X_3}(0) + f_{1+2}(s-2) f_{X_3}(2) + f_{1+2}(s-4) f_{X_3}(4)$$

x	$f_1(x)$	$f_2(x)$	$f_1 * f_2(x) = f_{1+2}(x)$	$f_3(x)$	$f_{1+2} * f_3(x) = f_{1+2+3}(x)$	$F_{1+2+3}(x)$
0	0.4	0.2	0.08	0.2	0.016	0.016
1	0.2	0	0.04	0	0.008	0.024
2	0.4	0.8	0.4	0.6	0.128	0.152
3	0	0	0.16	0	0.056	0.208
4	0	0	0.32	0.2	0.32	0.528
5	0	0	0	0	0.104	0.632
6	0	0	0	0	0.272	0.904
7	0	0	0	0	0.032	0.936
8	0	0	0	0	0.064	1
	1	1	1	1	1	1

Group 6.

Let f_1, f_2 and f_3 be given p.m.f. corresponding the random variables X_1, X_2 and X_3 as follows

x	0	1	2	3	4
$f_1(x)$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$		
$f_2(x)$	$\frac{1}{6}$	0	$\frac{5}{6}$		
$f_3(x)$	$\frac{2}{6}$		$\frac{3}{6}$		$\frac{1}{6}$

1. Use convolution and excel to fund the c.d.f. of $S = X_1 + X_2 + X_3$,
2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$ corresponding to the the aggregate loss S.

Group 7.

Let f_1, f_2 and f_3 be given p.m.f. corresponding the random variables X_1, X_2 and X_3 as follows

x	0	1	2	3	4
$f_1(x)$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{7}$		
$f_2(x)$	$\frac{3}{7}$	0	$\frac{4}{7}$		
$f_3(x)$	$\frac{1}{7}$		$\frac{5}{7}$		$\frac{1}{7}$

1. Use convolution and excel to fund the c.d.f. of $S = X_1 + X_2 + X_3$,
2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$ corresponding to the the aggregate loss S.

Group 8.

Let X_i, for $i = 1,\ 2,\ 3$ be independent and identically distributed with the common c.d.f.

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad x \ge 1. \end{cases}$$

1. Use convolution and excel to fund the c.d.f. of $S = X_1 + X_2 + X_3$,
2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$ corresponding to the the aggregate loss $8S$.

Solution:

Using the solution of the problem 4, we obtain for $8S$:
$P_{0.70} = 8 \times 1.776 = 14.208$, $P_{0.75} = 8 \times 1.8529 = 14.823$, $P_{0.80} = 8 \times 1.9371 = 15.497$, $P_{0.85} = 8 \times 2.0345 = 16.276$, $P_{0.90} = 8 \times 2.1566 = 17.253$, and $P_{0.85} = 8 \times 2.3306 = 18.645$.

Group 9.

Consider three independent random variables X_1, X_2, X_3. For $k = 1,\ 2,\ 3$, X_k has an exponential distribution and $E[X_k] = \dfrac{1}{k}$.

1. Derive by the convolution process the p.d.f. and c.d.f. of $S = X_1 + X_2 + X_3$.
2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$ corresponding to the the aggregate loss S.

Solution:

$$f_{1+2}(s) = f_{X_1+X_2}(s) = \int_{-\infty}^{+\infty} f_{X_1}(s-y)\, f_{X_2}(y)dy = \int_0^s e^{-(s-y)} 2e^{-2y} dy = 2e^{-s} \int_0^s e^{-y} dy = 2e^{-s}\left(1 - e^{-s}\right).$$

and

$$
\begin{aligned}
f_S(s) &= f_{1+2+3}(s) = \int_{-\infty}^{+\infty} f_{1+2}(s-y)\, f_3(y)dy = \int_0^s 2e^{-(s-y)}\left(1-e^{-(s-y)}\right)3e^{-3y}dy \\
&= 6\int_0^s e^{-(s-y)}e^{-3y}dy - 6\int_0^s e^{-2(s-y)}e^{-3y}dy = \frac{6e^{-s}}{2}\left(1-e^{-2s}\right) - 6e^{-2s}\left(1-e^{-s}\right) f_S(s) = 3e^{-s}\left(1-e^{-s}\right)^2
\end{aligned}
$$

$$
F_S(s) = \int_0^s f_S(t)dt = \int_0^s 3e^{-t}\left(1-e^{-t}\right)^2 dt = \left(1-e^{-s}\right)^3 \text{ for all } s \geq 0.
$$

Therefore the solution of the equation $F_S(P_\alpha) = \alpha$ is equivalent to $1-\alpha^{\frac{1}{3}} = e^{-P_\alpha}$ hence $P_\alpha = -\ln\left(1-\alpha^{\frac{1}{3}}\right)$.

Finally

$$
\begin{aligned}
P_{0.70} &= -\ln\left(1-(0.70)^{\frac{1}{3}}\right) = 2.1884, \quad P_{0.75} = -\ln\left(1-(0.75)^{\frac{1}{3}}\right) = 2.3921, \\
P_{0.80} &= -\ln\left(1-(0.80)^{\frac{1}{3}}\right) = 2.6355, \quad P_{0.85} = -\ln\left(1-(0.85)^{\frac{1}{3}}\right) = 2.9425, \\
P_{0.90} &= -\ln\left(1-(0.90)^{\frac{1}{3}}\right) = 3.3665, \quad P_{0.95} = -\ln\left(1-(0.95)^{\frac{1}{3}}\right) = 4.0773.
\end{aligned}
$$

Group 10.

Consider three independent random variables X_1, X_2, X_3. For $k = 1, 2, 3$, X_k has a Poisson distribution such that $E[X_k] = k$.

1. Derive by the convolution process the p.d.f. and c.d.f. of $S = X_1 + X_2 + X_3$.
2. Calculate the quantile percentile premiums P_α for $\alpha = 0.70, 0.75, 0.80, 0.85, 0.90, 0.95$ corresponding to the the aggregate loss S.

Group 11.

A fire insurance company covers 200 contracts against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Class k	1	2	3	4	5
Contract amount in thousand	2	3	4	5	6
Number of contracts	80	40	20	30	30
Probability of occurrence of the claim	0.11	0.11	0.11	0.11	0.11

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to the **contract amount** and the parameter of the exponential density is **11**. Let S be the amount aggregate claims in a 1–year period.

1. Calculate the mean and variance of S.
2. Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the $100\alpha^{th}$ percentile of the distribution of total claims for $\alpha = 0.70, 0.75, 0.80, 0.85, 0.90, 0.95$.

Group 12.

A fire insurance company covers 200 contracts against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Class k	1	2	3	4	5
Contract amount in thousand	2	3	4	5	6
Number of contracts	80	40	20	30	30
Probability of occurrence of the claim	0.12	0.12	0.12	0.12	0.12

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to the **contract amount** and the parameter of the exponential density is **12**. Let S be the amount aggregate claims in a 1–year period.

1. Calculate the mean and variance of S.
2. Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the $100\alpha^{th}$ percentile of the distribution of total claims for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$.

Group 13.

A fire insurance company covers 200 contracts against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Class k	1	2	3	4	5
Contract amount in thousand	2	3	4	5	6
Number of contracts	80	40	20	30	30
Probability of occurrence of the claim	0.13	0.13	0.13	0.13	0.13

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to the the **double** of the **contract amount** and the parameter of the exponential density is **13**. Let S be the amount aggregate claims in a 1–year period.

1. Calculate the mean and variance of S.
2. Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the $100\alpha^{th}$ percentile of the distribution of total claims for $\alpha = 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95$.

Group 3

A reinsurance company covers 200 contracts against the damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Lambda	Q1) Calculate the mean and variance of S.			
11	E(S)	2.00000	Var(S)	0.344
Coeff				
1				

$\frac{1}{\lambda}(1 - e^{-\lambda M})$

$\frac{1}{\lambda^2} - \frac{2M}{\lambda}e^{-\lambda M} - \frac{1}{\lambda^2}e^{-2\lambda M}$

$\mu_k^2 q_k(1 - q_k) + \sigma_k^2 q_k$

Class	Contract amount in thousand	Number of contracts	Probability of occurrence of the claim	μ	$E(X_k)$	μ^2	σ^2	$Var(X_k)$
1	2	80	0.11	0.090909	0.800000	0.008264	0.008264	0.137455
2	3	40	0.11	0.090909	0.400000	0.008264	0.008264	0.068727
3	4	20	0.11	0.090909	0.200000	0.008264	0.008264	0.034364
4	5	30	0.11	0.090909	0.300000	0.008264	0.008264	0.051545
5	6	30	0.11	0.090909	0.300000	0.008264	0.008264	0.051545
		200						

Assume that the conditional distribution of the claim size, given that a claim has occurred, follows a truncated exponential distribution over the interval from 0 to the contract amount and the parameter of the exponential density is 11. Let S be the amount aggregate claims in a 1 year period.

Q2) Use a normal approximation to calculate the safety loading premiums to be requested so the company can collect an amount equal to the 100α

α	ϕ	θ	Π_θ
0.70	0.524401	0.1537	2.3074
0.75	0.674490	0.1977	2.3954
0.80	0.841621	0.2467	2.4934
0.85	1.036433	0.3038	2.6076
0.90	1.281552	0.3756	2.7513
0.95	1.644854	0.4821	2.9642