King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2020–2021
Academic Year (H) 1442
Bachelor AFM: M. Eddahbi

April 1, 2021 from 3 to 5 PM, ACTU-462-472

Model Answer

Midterm Exam 2 Actuarial Mathematical Models 2

Exercise 1 ***(6 marks)***
For a special fully continuous last survivor insurance of 20000 on (x) and (y), you are given: (i) T_x and T_y are independent. (ii)

$$\mu_{x+t} = 0.04,\ t > 0 \qquad \delta = 0.03$$
$$\mu_{y+t} = \begin{cases} 0.02 & 0 < t < 20 \\ 0.04 & t \geq 20 \end{cases} \qquad Z_1 = 20000 v^{T_{\overline{xy}}}$$

The actuary has recommended to calculate the premium P of this insurance using the principle

$$P = \mathrm{E}[Z_1] + 2\%\sqrt{\mathrm{Var}\,(Z_1)}.$$

Find P. (Hint: $\bar{A}_{\overline{xy}} + \bar{A}_{xy} = \bar{A}_x + \bar{A}_y$, ${}^2\bar{A}_{xy} = {}^2\bar{A}^{\,1}_{xy:\overline{n}|} + {}^2{}_nE_{xy}\ {}^2\bar{A}_{x+n:y+n}$ and ${}^2{}_nE_{xy} = v^{2n}p_{xy}$). The symbol ^{2} means to replace δ by 2δ in $\bar{A}^{\,1}_{x:\overline{n}|}$, ${}_nE_x$, $\bar{A}^{\,1}_{xy:\overline{n}|}$ and ${}_nE_{xy}$.

Solution: By definition $\mathrm{E}[Z_1] = 20000\ \bar{A}_{\overline{xy}}$ and $\mathrm{E}[Z_1^2] = 4\times 10^8\ {}^2\bar{A}_{\overline{xy}}$. Also by the symmetric relation $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$ but

$$\begin{aligned} \bar{A}_x &= \frac{\mu_x}{\mu_x+\delta} = \frac{0.04}{0.04+0.03} = \frac{4}{7} = 0.57143, \\ {}^2\bar{A}_x &= \frac{0.04}{0.04+2(0.06)} = 0.25 \end{aligned}$$

$$\bar{A}_y = \bar{A}^{\,1}_{y:\overline{20}|} + {}_{20}E_y\ \bar{A}_{y+20} = \frac{0.02}{0.02+0.03}\left(1-e^{-0.05\times 20}\right) + e^{-0.05\times 20}\frac{4}{7} = 0.463065.$$

$$\begin{aligned} {}^2\bar{A}_y &= {}^2\bar{A}^{\,1}_{y:\overline{20}|} + {}^2{}_{20}E_y\ {}^2\bar{A}_{y+20} \\ &= \frac{0.02}{0.02+2(0.03)}\left(1-e^{-0.08\times 20}\right) + e^{-0.08\times 20}\times 0.463065 = 0.293017. \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{A}_{xy} &= \bar{A}^{\,1}_{xy:\overline{20}|} + {}_{20}E_{xy}\ \bar{A}_{x+20:y+20} \\ &= \frac{0.12}{0.12+0.06}\left(1-e^{-0.18\times 20}\right) + e^{-0.18\times 20}\frac{0.16}{0.16+0.06} \\ &= \frac{2}{3} + \frac{2}{33}e^{-3.6} = 0.668323 \\ {}^2\bar{A}_{xy} &= {}^2\bar{A}^{\,1}_{xy:\overline{20}|} + {}^2{}_{20}E_{xy}\ {}^2\bar{A}_{x+20:y+20} \\ &= \frac{0.06}{0.06+2(0.03)}\left(1-e^{-0.12\times 20}\right) + e^{-0.12\times 20}\frac{0.08}{0.08+2(0.03)} = 0.50648 \end{aligned}$$

By symmetric relation,

$$\bar{A}_{\overline{xy}} = 0.57143 + 0.463065 - 0.668323 = 0.366172, \quad {}^2\bar{A}_{\overline{xy}} = 0.1555265.$$

So,

$$\mathrm{Var}\,(Z_1) = 0.1555265 - (0.3263063)^2 = 0.04905.$$

Finally $P = 20000 \times 0.668323 + 0.02 \times 2 \times 10^4 \times \sqrt{0.04905} = \mathbf{13455}$.

Exercise 2 ***(6 marks)***

1. ***(3 marks)*** *For* $i = 0.06$, $\ell_{30} = 95014$, $\ell_{50} = 89510$, $\ddot{a}_{30:40} = 14.2068$ *and* $\ddot{a}_{40:50} = 12.4784$, *calculate* **APV** *of a* 10*–year temporary life annuity–immediate on independent lives* (30) *and* (40). *(Hint:* $a_{x:y:\overline{n}|} = a_{x:y} - {}_nE_{x:y}\ a_{x+n:y+n}$ *and* $a_{x:y} + 1 = \ddot{a}_{x:y}$*).*
2. ***(3 marks)*** *Let* Z *denotes the present–value random variable for a discrete whole life insurance of* 1 *issued to* (x) *and* (y) *which pays* 2000 *at the first death and* 2000 *at the second death. Given: (i)* $a_x = 9$ *(ii)* $a_y = 13$ *(iii)* $i = 0.04$. *Calculate the Net Single Premium for this policy. (Hint:* $A_{xy} + A_{\overline{xy}} = A_x + A_y$ *and* $A_z = 1 - d\ddot{a}_z$*).*

Solution:

1. We use the relation between annuity due and annuity immediate:

$$\begin{aligned} a_{30:40:\overline{10}|} &= a_{30:40} - {}_{10}E_{30:40}\ a_{40:50} \\ &= (\ddot{a}_{30:40} - 1) - v^{10}\frac{\ell_{40}}{\ell_{30}}\frac{\ell_{50}}{\ell_{40}}(\ddot{a}_{40:50} - 1) \\ &= 13.2068 - (1.06)^{-10}\frac{89510}{95014} \times 11.4784 = \mathbf{7.16861} \end{aligned}$$

2. The Net Single Premium for this policy denoted by α is the APV of the insurance and it is given by $\alpha := 2000\,(A_{xy} + A_{\overline{xy}})$ which can be written using symmetric relation as $\alpha = 2000\,(A_x + A_y)$. We have $\ddot{a}_x = 10$ and $\ddot{a}_y = 14$, and hence $A_x = 1 - d\ddot{a}_x = 1 - \frac{0.04}{1.04} \times 10 = 0.615385$ and $A_y = 1 - d\ddot{a}_y = 1 - \frac{0.04}{1.04} \times 14 = 0.461538$. Therefore $\alpha = 2000\,(0.615385 + 0.461538) = \mathbf{2153.846}$.

Exercise 3 ***(6 marks)***

1. ***(3 marks)*** *Two independent lives* (30) *and* (40) *purchase a joint life insurance that pays* 50000 SAR *at the end of the year of the first death. Level premium is payable at the beginning of the year until the first death. Given* $i = 0.06$, $\ddot{a}_{30:40} = 14.2068$ *and* $\ddot{a}_{35:45} = 13.4150$, *calculate the time–*5 *net premium reserve for the policy.(Hint:* $A_{x:y} = 1 - d\ddot{a}_{x:y}$*).*
2. ***(3 marks)*** *Assume that the mortality for two independent lives aged* (70) *and* (75) *follows De Moivre's law with* $\omega = 100$. *Calculate* $10000\bar{A}_{\overset{1}{70}:75}$ *given* $\bar{a}_{75} = 8.655$. *(Hint: notice that* ${}_tp_{75} = 0$ *for* $t > 25$*).*

Solution:

1. The level annual premium for the unit benefit is given by

$$P_{30:40} = \frac{A_{30:40}}{\ddot{a}_{30:40}} = \frac{1 - d\ddot{a}_{30:40}}{\ddot{a}_{30:40}} = \frac{1 - \frac{0.06}{1.06}14.2068}{14.2068} = 0.013785$$

The time–5 reserve is

$$\begin{aligned} 50000\ {}_5V_{35:45} &= 50000\,(A_{35:45} - P_{30:40}\ \ddot{a}_{35:45}) \\ &= 50000\,(1 - d\ddot{a}_{35:45} - P_{30:40}\ \ddot{a}_{35:45}) \\ &= 50000\left(1 - \frac{0.06}{1.06}13.4150 - 0.013785 \times 13.4150\right) \\ &= \mathbf{2786.73012} \end{aligned}$$

2. By definition we have

$$\bar{A}_{\overset{1}{70}:75} = \int_0^{30} v^t\ {}_tp_{75}\ \mathrm{f}_{70}(t)\mathrm{d}t$$

but $\mathrm{f}_{70}(t) = \dfrac{1}{30}$ for $0 < t < 30$ and $\mathrm{f}_{70}(t) = 0$ otherwise. Also T_{75} is uniformly distributed in $(0, 25)$, thus ${}_tp_{75} = 0$ for $t > 25$, consequently

$$\bar{A}_{\overset{1}{70}:75} = \frac{1}{30}\int_0^{25} v^t\ {}_tp_{75}\ \mathrm{d}t = \frac{1}{30}\bar{a}_{75} = \frac{8.655}{30} = 0.2885.$$

Finally $10000\bar{A}_{\overset{1}{70}:75} = \mathbf{2885}$.

Exercise 4 ***(6 marks)***
For a multiple state model of an insurance on (x) and (y): (i) The death benefit of 10000 SAR is payable at the moment of the second death. (ii) You use the states: State 0 = both alive State 1 = only (x) is alive State 2 = only (y) is alive State 3 = neither alive (iii) $\mu^{01}_{x+t:y+t} = \mu^{02}_{x+t:y+t} = 0.03$, $t \geq 0$ (iii) $\mu^{03}_{x+t:y+t} = 0$, $t \geq 0$ (iv) $\mu^{13}_{x+t:y+t} = \mu^{23}_{x+t:y+t} = 0.04$, $t \geq 0$ (v) $\delta = 0.05$. Calculate the expected present value of this insurance on (x) and (y) given the following information

$${}_tp^{02}_{xy} = {}_tp^{01}_{xy} = \frac{3}{2}\left(e^{-0.04t} - e^{-0.06t}\right).$$

Solution: The model in the question is the model with no common shock component. We use

$$\bar{A}_{\overline{xy}} = \int_0^{\infty} v^t\left({}_tp^{01}_{xy}\ \mu^{13}_{x+t} + {}_tp^{02}_{xy}\ \mu^{23}_{y+t}\right)\mathrm{d}t = 2\int_0^{\infty} e^{-\delta t}\ {}_tp^{01}_{xy}\ \mu^{13}_{x+t}\ \mathrm{d}t,$$

Observe that

$$\begin{aligned} {}_tp^{02}_{xy} &= {}_tp^{01}_{xy} = \int_0^t {}_sp^{00}_{xy}\ \mu^{01}_{x+s:y+s}\ {}_{t-s}p^{11}_{x+s}\ \mathrm{d}s = \int_0^t e^{-0.06s}0.03e^{-0.04(t-s)}\mathrm{d}s \\ &= 0.03e^{-0.04t}\int_0^t e^{-0.02s}\ \mathrm{d}s = \frac{3}{2}e^{-0.04t}\left(1 - e^{-0.02t}\right) = \frac{3}{2}e^{-0.04t} - \frac{3}{2}e^{-0.06t} \end{aligned}$$

and hence

$$\begin{aligned}\bar{A}_{\overline{xy}} &= 2\int_0^{\infty} e^{-0.05t}\frac{3}{2}\left(e^{-0.04t}-e^{-0.06t}\right)0.04\mathrm{d}t = \frac{12}{100}\int_0^{\infty}\left(e^{-0.09t}-e^{-0.11t}\right)\mathrm{d}t\\ &= 12\left(\frac{1}{9}-\frac{1}{11}\right) = \frac{8}{33} = 0.24242424.\end{aligned}$$

The final result is $10000\bar{A}_{\overline{xy}} = 10000 \times 0.24242424 = \mathbf{2424.2424}$.

Exercise 5 ***(6 marks)***

1. *(**2 marks**) You are given the following information:*
   (i) A zero–coupon \$100 par–value one–year government bond is trading at a price of \$96. (The par–value is the amount of money that bond issuers promise to repay bond–holders at the maturity date of the bond.)
   (ii) A two–year \$1000, 6% annual coupon government bond is trading at a price of \$1020. Let $f(t, t+k)$ be the forward interest rate, contracted at time 0, effective from time t to $t+k$. Find $f(1,2)$.
   Hint:
   $$v(t) = \frac{1}{(1+y_t)^t} \quad \text{and} \quad (1+f(t,t+k))^k = \frac{(1+y_{t+k})^{t+k}}{(1+y_t)^t} = \frac{v(t)}{v(t+k)}.$$
   where $v(t)$ is the current price of zero–coupon bond that pays \$1 with certainty t years from now, y_t is the t–year spot interest rate.

2. *(**4 marks**) The price of a \$100 zero–coupon bond maturing one year from now is now trading at a price of \$96. The price of a \$100, 5% annual coupon bond with exactly two years to maturity is now trading at a price of \$107. The price of a \$100, 7% annual coupon bond with exactly three years to maturity is now trading at a price of \$111. Assuming that $\mu_x = 0.04$ for x, calculate the net single premium of a 3–year term life insurance on (x) that pays \$30000 at the end of the year of death. (Hint: The present value of notional N and r% annual coupon bond with exactly n years to maturity is $PV = N[r\sum_{k=1}^{n-1} v(k) + (1+r)v(n)]$ **bootstrap method**).*

Solution

1. From (i), we can immediately obtain the 1–year spot rate as $y_1 = \frac{100}{96} - 1 = \frac{1}{24} = 0.0416667$. From (ii) we have
   $$\frac{60}{1+y_1} + \frac{1060}{(1+y_2)^2} = \frac{60}{1+\frac{1}{24}} + \frac{1060}{(1+y_2)^2} = 1020$$
   which gives $y_2 = 0.0494823$, in fact we need just $(1+y_2)^2 = 1.101413134$ now we shall use $(1+f(1,2)) = \frac{(1+y_2)^2}{1+y_1}$ to back up $f(1,2) = \frac{1.101413134}{0.0416667} - 1 = \mathbf{0.057356}$.
   One can also calculate $f(1,2)$ by using the equation below:
   $$\frac{60}{1+y_1} + \frac{1060}{(1+y_1)(1+f(1,2))} = \frac{60}{1+\frac{1}{24}} + \frac{1060}{\left(1+\frac{1}{24}\right)(1+f(1,2))} = 1020$$
   $f(1,2) = \frac{23}{401} = \mathbf{0.057356}$.

2. The net single premium can be expressed as

$$\text{NSP} = 30000\left(v(1)\ q_x + \nu(2)\ p_x\ q_{x+1} + v(3)\ {}_2p_x\ q_{x+2}\right).$$

First, we calculate the values of $v(1)$, $v(2)$ and $v(3)$. The value of $v(1)$ is simply $\frac{96}{100} = 0.96$. The values of $v(2)$ and $v(3)$ are not given and have to be obtained by the **bootstrap method**. We have

$$5v(1) + 105v(2) = 107$$

which gives $v(2) = 0.973333$. Also, we have

$$7v(1) + 7v(2) + 107v(3) = 111$$

which gives $v(3) = 0.910903$.
Second, we calculate the death and survival probabilities. Since $\mu_x = 0.04$ for all $x \geq 0$, we have

$$ {}_tp_x = e^{-0.04t} \text{ and } {}_tq_x = 1 - e^{-0.04t}.$$

As a result, the net single premium is given by

$$30000\left(1 - e^{-0.04}\right)\left(0.96 + 0.973333e^{-0.04} + 0.910903e^{-0.08}\right) = \mathbf{3218.44696}.$$