King Saud University
College of Sciences
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Model Answer of the Final exam ACTU–462 (40%)

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Problem 1. (9 marks)

1. For a special fully continuous whole life insurance on (65), the death benefit at time t is $b_t = 1000e^{0.04t}$, for $t \geq 0$, level premiums are payable for life and $\mu_{65+t} = 0.02$, $t \geq 0$ and $\delta = 0.04$.
   (a) **(3 marks)** Calculate the annual net premium for this life insurance.
   (b) **(3 marks)** Calculate the premium reserve at the end of year 2.
2. **(3 marks)** For a fully discrete whole life insurance of 1000 on (50), you are given: $1000P_{50} = 25$, $1000A_{61} = 440$, $1000q_{60} = 20$, and $i = 6\%$. Calculate $1000\ {}_{10}V_{50}$.

Solution:

1.
   (a) $\text{APV(F.B.)}_0 = 1000$ and $\text{APV(F.P.)}_0 = P\bar{a}_{65}$ where

$$\bar{a}_{65} = \int_0^\infty e^{-\delta t}\ {}_tp_{65} dt = \int_0^\infty e^{-(\delta+\mu)t} dt = \frac{1}{\delta+\mu} = \frac{1}{0.06} = \frac{50}{3}.$$

Hence by the equivalence principle

$$1000 = \frac{50}{3}P = 0 \iff P = \frac{3000}{50} = \mathbf{60}$$

   (b) The APV, at time 2, of future benefit is

$$\begin{aligned}\int_0^\infty b_{t+2} v^t\ {}_tp_{67}\ \mu_{67+t} dt &= \int_0^\infty 1000e^{0.04(t+2)} e^{-0.04t}\ e^{-0.02t} 0.02 dt \\ &= 1000e^{0.04\times 2} \int_2^\infty e^{-0.02t} 0.02 dt = 1000e^{0.08} = 1083.3\end{aligned}$$

and the APV, at time 2, of future premiums is $P\bar{a}_{67} = P\bar{a}_{65}$ (thanks to the CFM assumption). Therefore

$${}_2V = 1083.3 - 60 \times \frac{50}{3} = \mathbf{83.3}$$

2. We know that

$$\begin{aligned}1000\ {}_{10}V_{50} &= 1000\left(A_{60} - P_{50}\ddot{a}_{60}\right) = 1000A_{60} - 1000P_{50}\ddot{a}_{60} \\ &= 1000\ A_{60} - 25\ddot{a}_{60} = 1000A_{60} - 25\left(\frac{1-A_{60}}{d}\right).\end{aligned}$$

Now, we need A_{60} By recursion relation for life insurance we can write

$$A_{60} = vq_{60} + vp_{60}\ A_{61},$$

then

$$\begin{aligned} 1000\ A_{60} &= v \times 1000\ q_{60} + vp_{60} \times 1000\ A_{61} \\ &= \frac{20}{1.06} + \frac{1-0.02}{1.06} \times 440 = 425.66 \end{aligned}$$

Consequently

$$1000\ {}_{10}V_{50} = 425.66 - 25\left(\frac{1-0.42566}{0.06}\right)1.06 = \mathbf{171.99}$$

Problem 2. (9 marks)

1. For a fully discrete 20–year deferred whole life insurance of 1000 on (50), such that Premiums are payable for 20 years and Deaths are Uniformly Distributed between integral ages. Given $i = 0.045$, $q_{59} = q_{70} = 0.5$ and ${}_9V = 60$, ${}_{9.5}V = 250$, ${}_{20.5}V = 1850$.
   (3 marks) Calculate the level net premium for this policy.
2. **(3 marks)** Calculate ${}_{10}V$, the net premium reserve at the end of year 10.
3. **(3 marks)** Calculate ${}_{21}V$, the net premium reserve at the end of year 21.

Solution:

1. From recursion formula we have

$$\begin{aligned} ({}_9V + P)(1+i)^{0.5} &= v^{1-s}\ {}_{0.5}q_{59} \times 0 +\ {}_{9.5}V\ {}_{0.5}p_{59} \\ &= (1 -\ {}_{0.5}q_{59})\ {}_{9.5}V = \left(1 - \frac{1}{2} \times \frac{1}{2}\right)250 = 187.5 \end{aligned}$$

   then

$$P = \frac{187.5}{\sqrt{1.045}} - 60 = \mathbf{123.418}.$$

2. From recursion formula we have also

$$({}_{9.5}V + 0)\sqrt{1.045} = {}_{10}V\ {}_{0.5}p_{59.5} = {}_{10}V\frac{2}{3}.$$

   since ${}_{0.5}p_{59.5} = \frac{p_{59}}{{}_{0.5}p_{59}} = \frac{p_{59}}{1-\ 0.5q_{59}} = \frac{0.5}{1-0.25} = \frac{2}{3} = 0.66667$, hence

$${}_{10}V = \frac{3}{2}{}_{9.5}V\sqrt{1.045} = \frac{3}{2}250 \times \sqrt{1.045} = \mathbf{383.34}$$

3. Observe first that ${}_{0.5}p_{70.5} = \frac{p_{70}}{{}_{0.5}p_{70}} = \frac{p_{70}}{1-\ 0.5q_{70}} = \frac{2}{3} = 0.66667$, therefore

$${}_{20.5}V\sqrt{1.045} = {}_{0.5}q_{70.5} \times 1000 +\ {}_{21}V\ {}_{0.5}p_{70.5} = \frac{1000}{3} + \frac{2}{3}\ {}_{21}V,$$

   then

$$\begin{aligned} {}_{21}V &= \left({}_{20.5}V\sqrt{1.045} - \frac{1000}{3}\right)\frac{3}{2} \\ &= \left(1850\sqrt{1.045} - \frac{1000}{3}\right)\frac{3}{2} = \mathbf{2336.8}. \end{aligned}$$

<u>Problem 3.</u> (9 marks)

1. **(3 marks)** For a fully discrete, 2–payment, 3–year term insurance of 20,000 on (x), you are given: (i) $i = 0.10$ $q_x = 0.2$, $q_{x+1} = 0.25$, $q_{x+2} = 0.5$. (ii) Expenses, paid at the beginning of the policy year, are:

	Per Policy	Per 1000 of Insurance	Fraction of Premium
First Year	50	4.50	0.18
Second Year	15	1.50	0.10
Third Year	15	1.50	–

   (iv) Settlement expenses, paid at the end of the year of death, are 30 per policy plus 1 per 1000 of insurance.

   Calculate the gross premium reserve for this insurance at time 1.

2. **(3 marks)** For a fully discrete whole life policy of 1000 issued to (65): Mortality follows the Illustrative Life Table and $i = 0.06$. Calculate the **first year modified premium**, **renewal modified premium** under the **full preliminary term method**, and calculate **the reserve at the end of year** 5.

3. **(3 marks)** For a fully continuous 20–year deferred whole life insurance of 10,000 on (45), you are given: (i) $\bar{A}_{65} = 0.25821$ (ii) The annual net premium is 71.25, and is payable for the first 20 years. (iii) $\mu_x = 0.00015(1.06)^x$ and $\delta = 0.05$. Use Euler's method with step 0.5 to calculate ${}_{19}V$.

Solution:

1. We have first to find the gross premium G,

$$\text{APV(FP)}_0 = G(1 + vp_x) = G(1 + \frac{1 - 0.2}{1.1}) = 1.7273G.$$

$$\begin{aligned}
\text{APV(FB + FE)}_0 &= (20,000 + 30 + 20)A^{1}_{x:\overline{3}|} + 50 + 4.5(20) + 0.18G + (15 + 1.5(20))\, a_{x:\overline{2}|} + 0.1Gvp_x \\
&= (20050)A^{1}_{x:\overline{3}|} + 140 + 0.18G + 45a_{x:\overline{2}|} + 0.1Gvp_x \\
&= (20050)A^{1}_{x:\overline{3}|} + 140 + 0.18G + 45\left(vp_x + v^2\ {}_2p_x\right) + 0.1Gvp_x
\end{aligned}$$

Moreover

$$\begin{aligned}
A^{1}_{x:\overline{3}|} &= vq_x + v^2 p_x q_{x+1} + v^3\ {}_2p_x q_{x+2} = vq_x + v^2(1 - q_x)\, q_{x+1} + v^3\ (1 - q_x)(1 - q_{x+1})\, q_{x+2} \\
&= \frac{0.2}{1.1} + \frac{1}{1.1^2}(1 - 0.2)\, 0.25 + \frac{1}{1.1^3}(1 - 0.2)(1 - 0.25)\, 0.5 = 0.5725.
\end{aligned}$$

and

$$vp_x + v^2\ {}_2p_x = \frac{0.8}{1.1} + \frac{0.8}{1.1^2}0.75 = 1.2231.$$

Therefore

$$\begin{aligned}
\text{APV(FB + FE)}_0 &= 20050 \times 0.5725 + 140 + 0.18G + 45 \times 1.2231 + 0.1G\frac{0.8}{1.1} \\
&= 0.25273G + 11674.
\end{aligned}$$

Now by E.P. $1.7273G = 0.25273G + 11674$, thus $G = 7916.82$.
At time 1, the APV of future benefits plus settlement expenses is

$$20050\left(\frac{0.25}{1.1}+\frac{(0.75)(0.5)}{1.1^2}\right)=10770.66.$$

The APV of future renewal per–policy and per–1000 expenses is

$$0.1\times 7916.82+45(1+\frac{0.75}{1.1})=867.36.$$

So the gross premium reserve is

$${}_1V^g = 10770.66+867.36-7916.82=\mathbf{3721.2}.$$

2. The modified premium in the first year is $1000\alpha = 1000vq_{65} = \frac{21.32}{1.06} = \mathbf{20.11321}$. The modified premium in renewal years is the net premium for (66), or

$$1000\beta = 1000P_{66} = 1000\frac{A_{66}}{\ddot{a}_{66}} = \frac{454.56}{9.6362} = \mathbf{47.17212}$$

The reserve at the end of 5 years is the net premium reserve at time 4 for a whole life issued on (66), which is given by

$${}_5V^{\mathrm{FPT}} = 1000\ {}_4V_{66} = 1000\left(1-\frac{\ddot{a}_{70}}{\ddot{a}_{66}}\right) = 1000\left(1-\frac{8.5693}{9.6362}\right) = \mathbf{110.72}$$

3. There is no death benefit in year 19, so the benefit $b_t = 0$ in that period. Let us calculate the two μ_x's that we need.

$$\mu_{64.5} = 0.00015(1.06)^{64.5} = 0.00643161 \quad \text{and} \quad \mu_{64} = 0.00015(1.06)^{64} = 0.00624693$$

The net premium reserve at time 20, since the policy is paid up then, is $10000\bar{A}_{65} = 2582.10$. We shall apply the discritization

$${}_tV^g \simeq \frac{{}_{t+h}V^g - h\left(G_t-(e_t+c_tG_t)-(b_t+E_t)\,\mu_{x+t}\right)}{1+h\left(\delta_t+\mu_{x+t}\right)}.$$

Since ${}_{20}V = 2582.10$

$$\begin{aligned} {}_{19.5}V &= \frac{2582.10-0.5(71.25)}{1+0.5(0.05+0.00643161)} = \mathbf{2476.60}. \\ {}_{19}V &= \frac{2476.60-0.5(71.25)}{1+0.5(0.05+0.00624693)} = \mathbf{2374.20}. \end{aligned}$$

Problem 4. **(9 marks)**

1. **(3 marks)** For a 3–year fully discrete term insurance of 1000 on (40) subject to a double decrement model such that: Decrement 1 is death. Decrement 2 is withdrawal. There are no withdrawal benefits, $i = 0.05$ and

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
40	2000	20	60
41	–	30	50
42	–	40	–

(a) Calculate the level annual net premium for this insurance.

(b) Calculate ${}_2V$, the net premium reserve at the end of year 2.

2. **(3 marks)** For a special fully continuous whole life insurance of 1 on (x), you are given: (i) Mortality follows a double decrement model. (ii) The death benefit due to cause 1 is 30000 and the death benefit due to cause 2 is 10000. (iii) $\mu_{x+t}^{(1)} = 0.02$, $\mu_{x+t}^{(2)} = 0.04$ for any $t \geq 0$ and the force of interest, δ, is a positive constant. Calculate the net premium for this insurance.

3. **(3 marks)** For a special insurance on (x) there are three causes of decrement such that

j	1	2	3
$\mu_{x+t}^{(j)}$	0.005	0.010	0.020
$b_{x+t}^{(j)}$	$\begin{cases} 3 \times 10^5 & \text{if } 0 \leq t \leq 10, \\ 0 & \text{if } t > 10. \end{cases}$	$\begin{cases} 2 \times 10^5 & \text{if } 0 \leq t \leq 20, \\ 0 & \text{if } t > 20. \end{cases}$	10^5 for any $t \geq 0$

Benefits are payable at the moment of decrement and $\delta = 4\%$. Calculate the single net premium for this insurance.

Solution:

1.

(a) $\ell_{41}^{(\tau)} = 2000 - 20 - 60 = 1920$ and $\ell_{42}^{(\tau)} = 1920 - 30 - 50 = 1840$. The actuarial present value of the death benefits is

$$1000\left(vq_{40}^{(1)} + v^2 p_{40}^{(\tau)} q_{41}^{(1)} + v^3\ {}_2p_{40}^{(\tau)} q_{42}^{(1)}\right) = \frac{1000}{2000}\left(\frac{20}{1.05} + \frac{30}{1.05^2} + \frac{40}{1.05^3}\right) = 40.4060$$

actuarial present value of the future premiums is

$$P\left(1 + vp_{40}^{(\tau)} + v^2\ {}_2p_{40}^{(\tau)}\right) = P\left(1 + \frac{1}{2000}\left(\frac{1920}{1.05} + \frac{1840}{1.05^2}\right)\right) = 2.74875P$$

So the net premium is $P = \frac{40.4060}{2.74875} = \mathbf{14.700}$.

(b) ${}_2V = vb_3\, q_{42}^{(1)} - P = 1000v\frac{d_{42}^{(1)}}{\ell_{42}^{(\tau)}} - P = \frac{1000}{1+i}\frac{d_{42}^{(1)}}{\ell_{40}^{(\tau)} - d_{40}^{(\tau)} - d_{41}^{(\tau)}} - P = \frac{1000}{1.05}\frac{40}{2000-80-80} - 14.700 = \mathbf{6.0039}$.

2. We know that

$$\begin{aligned} \text{APV(FB)}_0 &= 10^4 \int_0^\infty e^{-\delta t}\ {}_tp_x^{(\tau)}(3\mu_{x+t}^{(1)} + \mu_{x+t}^{(2)})dt = 10^4 \int_0^\infty e^{-(\delta+0.06)t}\ (3 \times 0.02 + 0.04)dt \\ &= 10^3 \int_0^\infty e^{-(\delta+0.06)t} dt = \frac{10^3}{\mu_x^{(\tau)} + \delta}. \end{aligned}$$

By E.P.

$$P = \frac{\text{APV(FB)}_0}{\bar{a}_x^{(\tau)}} = \frac{\frac{10^3}{\mu_x^{(\tau)}+\delta}}{\frac{1}{\mu_x^{(\tau)}+\delta}} = \mathbf{1000}.$$

3. The single net premium for this insurance is the APV of future benefits which is given by

$$\begin{aligned} \text{APV(FB)}_0 &= 10^5 \left(3 \int_0^{10} e^{-\delta t} \; {}_tp_x^{(\tau)} \mu_{x+t}^{(1)} dt + 2 \int_0^{20} e^{-\delta t} \; {}_tp_x^{(\tau)} \mu_{x+t}^{(2)} dt + \int_0^{\infty} e^{-\delta t} \; {}_tp_x^{(\tau)} \mu_{x+t}^{(3)} dt \right) \\ &= 10^5 \left(3 \int_0^{10} e^{-0.075t} 0.005 dt + 2 \int_0^{20} e^{-0.075t} 0.010 dt + \int_0^{\infty} e^{-0.075t} 0.020 dt \right) = \mathbf{57936}. \end{aligned}$$

Problem 5. (9 marks)

1. **(4 marks)** Consider a permanent disability model with three states: State 0: Healthy, State 1: Permanently disabled, and State 2: Dead. Suppose that $\mu_x^{01} = 0.02$, $\mu_x^{02} = 0.04$, $\mu_x^{12} = 0.05$ for $x \geq 0$. For a person who is healthy at age 50, calculate the probability that
   (a) he is healthy at age 60;
   (b) he cannot survive to age 60.

2. **(5 marks+4 bonus)** In a permanent disability model, $\mu_x^{01} = 0.05$, $\mu_x^{02} = 0.02$, $\mu_x^{12} = 0.03$ and $\delta = 0.06$.
   (a) Calculate $\bar{a}_x^{00}$, $\bar{a}_x^{11}$ and $\bar{a}_x^{01}$.
   (b) Calculate $\bar{A}_x^{12}$ and $\bar{A}_x^{02}$
   (c) A permanent disability insurance pays continuously at the rate of 1 per year while the insured is disabled. The policyholder pays continuous premiums while he is healthy. Calculate the annual premium rate using the equivalence principle.
   (d) Calculate the continuous premium payable annually for this insurance using the equivalence principle if:
      i. The premiums are payable in state 0 only.
      ii. The premiums are payable in both states 0 and state 1.

Solution:

1.
   (a) The total rate of exit from state 0 is $\nu_x^0 = \mu_x^{01} + \mu_x^{02} = 0.06$. We know that ${}_tp_{50}^{00} = e^{-0.06t}$, thus ${}_{10}p_{50}^{00} = e^{-0.6} = \mathbf{0.54881}$.
   (b) The total rate of exit from state 1 is 0.05. Similarly, ${}_tp_{50}^{11} = e^{-0.05t}$ for any $x \geq 0$ and $t \geq 0$. Now, from the formula sheet we have

$$\begin{aligned} {}_{10}p_{50}^{01} &= \int_0^{10} {}_sp_{50}^{00} \; \mu_{50+s}^{01} \; {}_{10-s}p_{50+s}^{11} ds = \int_0^{10} e^{-0.06s} \; 0.02 \; e^{-0.05(10-s)} ds \\ &= 0.02e^{-0.5} \int_0^{10} e^{-0.01s} ds = \frac{0.02e^{-0.5}}{0.01} \left(1 - e^{-0.1}\right) = 2e^{-0.5} \left(1 - e^{-0.1}\right). \end{aligned}$$

   So the probability that a person who is healthy at age 50 cannot survive to age 60 is given by

$$1 - {}_{10}p_{50}^{00} - {}_{10}p_{50}^{01} = 1 - e^{-0.6} - 2e^{-0.5}(1 - e^{-0.1}) = \mathbf{0.33575}.$$

2.

(a) The first annuity is like a single–decrement annuity with a decrement rate of $0.05 + 0.02 = 0.07$, so its APV is

$$\bar{a}_x^{00} = \int_0^\infty e^{-\delta t}\ {}_tp_x^{00} dt = \int_0^\infty e^{-\delta t} e^{-\nu^0 t} dt = \frac{1}{\mu_x^{01} + \mu_x^{02} + \delta} = \frac{1}{0.13} = \mathbf{7.6923}.$$

Similarly, the second annuity has decrement rate $\mu^{12} = 0.03$ and has APV equal to

$$\bar{a}_x^{11} = \int_0^\infty e^{-\delta t}\ {}_tp_x^{11} dt = \int_0^\infty e^{-\delta t} e^{-\mu^{12} t} dt \frac{1}{0.03 + 0.06} = \frac{100}{9} = \mathbf{11.1111}.$$

For the third annuity

$$\begin{aligned} {}_tp_x^{01} &= \int_0^t {}_sp_x^{00} \mu_{x+s}^{01}\ {}_{t-s}p_{x+s}^{11} ds = 0.05 \int_0^t e^{-0.07s} e^{-0.03(t-s)} ds = 0.05 e^{-0.03t} \int_0^t e^{-0.04s} ds \\ &= \frac{0.05}{0.04} e^{-0.03t} \left(1 - e^{-0.04t}\right) = \frac{5}{4}\left(e^{-0.03t} - e^{-0.07t}\right). \end{aligned}$$

Thus

$$\begin{aligned} \bar{a}_x^{01} &= \int_0^\infty e^{-\delta t}\ {}_tp_x^{01} dt = \frac{5}{4} \int_0^\infty e^{-0.06t} \left(e^{-0.03t} - e^{-0.07t}\right) dt = \frac{5}{4}\left(\int_0^\infty e^{-0.09t} dt - \int_0^\infty e^{-0.13t} dt\right) \\ &= \frac{5}{4}\left(\frac{1}{0.09} - \frac{1}{0.13}\right) = \mathbf{4.27350}. \end{aligned}$$

(b) $\bar{A}_x^{12}$ is a standard whole life insurance since there's only one path from state 1 to state 2. So it is $\frac{\mu^{12}}{\mu^{12}+\delta} = \frac{0.03}{0.03+0.06} = \frac{1}{3} = \mathbf{0.33333}$.
For $\bar{A}_x^{02}$ there are two paths from state 0 to state 2. The APV of the direct path is

$$\begin{aligned} \int_0^\infty e^{-\delta t}\ {}_tp_x^{00} \mu^{02} dt &= \int_0^\infty e^{-\delta t} e^{-\nu^0 t} \mu^{02} dt = \frac{\mu^{02}}{\nu^0 + \delta} = \frac{\mu^{02}}{\mu_x^{01} + \mu_x^{02} + \delta} \\ &= \frac{0.02}{0.05 + 0.02 + 0.06} = \frac{2}{13} = 0.15385. \end{aligned}$$

For the other path, we use the formula for ${}_tp_x^{01}$ that we developed previously. So the APV of the path $(0 \to 1 \to 2)$ is

$$\int_0^\infty e^{-\delta t}\ {}_tp_x^{01} \mu^{12} dt = 0.03 \bar{a}_x^{01} = 0.03 \times 4.27350 = 0.12821.$$

Thus $\bar{A}_x^{02} = 0.15385 + 0.12821 = \mathbf{0.28206}$.

(c) By the equivalence principle, $P\bar{a}_x^{00} = \bar{a}_x^{01}$, hence $P = \frac{4.2735}{7.6923} = \mathbf{0.55556}$.

(d)

i. For an annuity in state 0, the APV is 7.6923, so the premium is $\frac{0.282051}{7.6923} = \mathbf{0.036667}$.

ii. For an annuity payable in state 1 for someone currently in state 0, the APV is 4.2735. So the APV of an annuity payable whether in state 0 or 1 is the sum of an annuity payable in state 0 and an annuity payable in state 1, and the premium for the insurance is $\frac{0.282051}{7.6923+4.2735} = \mathbf{0.023571}$.