College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

Second Midterm Exam
Academic Year 1442-1443 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	توزيعات الخسارة		اسم المقرر
Course Code	466 ريك		رمز المقرر
Exam Date	2020-11-23	1442-04-08	تاريخ الامتحان
Exam Time	10: 00 AM		وقت الامتحان
Exam Duration	2 hours	ساعتان	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
Instructor Name	Khaled Bennour		اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 2 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
-

تعليمات عامة:

- عدد صفحات الامتحان 2 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
-

هذا الجزء خاص بأستاذ المادة
This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
3				
4				
5				
6				
7				
8				

EXAM COVER PAGE

Explain your reasoning for why you have answered a certain value.

Exercise 1 *You have the following sample of values:*

$$3, 3, 5, 5, 6, 7, 8, 8, 8, 12.$$

You make two estimates of the cumulative hazard rate function $H(x)$. $\hat{H}_1(x)$ *is the Nelson-Aalen estimate assuming that all the data points reflect uncensored values.* $\hat{H}_2(x)$ *is the Nelson-Aalen estimate assuming that all data points of* 5 *and* 8 *are actually points of right censoring. Find* $\left|\hat{H}_1(10) - \hat{H}_2(10)\right|$.
Response: 1

Exercise 2 *Losses from a Mutual Insurance Company are denoted by a random variable* X *that follows a Pareto distribution with* $\alpha = 3$ *and* $\theta = 1,000$. *The Mutual Insurance Company requires policyholders to have a franchise deductible of* 500. *Find the expected cost per loss under this policy.*

Exercise 3 *Losses in* 2016 *from falling cows follow an exponential distribution with mean* $\theta = 340$. *Falling Cow Insurance Company requires policyholders to have an ordinary deductible of* 200. *Losses in* 2017 *are uniformly* 30% *higher than in* 2016.
a) Show that losses in 2017 *follows an exponential distribution and find its mean.*
b) Deduce the expected cost per loss $E(Y^L)$ *in* 2017.

Exercise 4 *Dental claims for* 2008 *are distributed exponentially. During* 2008, *the company did not impose a deductible or upper limit. The variance of the claims was* 40,000. *Dental claims for* 2009 *increase uniformly for by* 20%. *For* 2009, *the company also imposes an ordinary deductible of* 50 *and an upper limit so that the most the company will pay on any claim is* 250. *Calculate the expected claim amount per loss event* $E(Y^L)$.
Hint: for a policy subject to an ordinary deductible d *and a policy limit* u_*, $Y^L = (X \wedge u) - (X \wedge d)$, *with* $u = u_* + d$.

Exercise 5 *For a portfolio of policies, you are given:*
(i) There is no deductible and the policy limit varies by policy.
(ii) A sample of ten claims is:
350 350 500 500 500^+ 1000 1000^+ 1000^+ 1200 1500
where the symbol + indicates that the loss exceeds the policy limit.

Determine $\hat{S}(1250)$ *the Kaplan-Meier product-limit estimate of* $S(1250)$.

Exercise 6 *Given the following grouped data.*

Claim size	*Number of claims*
$(0, 35]$	18
$(35, 95]$	230
$(95, 145]$	320
$(145, 300]$	120
$(300, 500]$	312

Assume a uniform distribution of claim sizes within each interval. Find the density function corresponding to this data set.

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1] \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1 \\
\mathrm{E}[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
\mathrm{E}[X \wedge x] &= -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k \left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
\mathrm{E}[X^k] &= \frac{\theta^k (-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \tau \int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k \left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1} - 1)^{-1/\gamma} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad \mathrm{E}[X^k] = \theta^k \Gamma(k+1), \quad k > -1 \\
\mathrm{E}[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= -\theta \ln(1-p) \\
\mathrm{TVaR}_p(X) &= -\theta \ln(1-p) + \theta \\
\mathrm{E}[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\
&= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}$$

A.3.3.2 Inverse exponential—θ

$$\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k G(1-k; \theta/x) + x^k(1 - e^{-\theta/x}), \quad \text{all } k \\
\text{mode} &= \theta/2
\end{aligned}$$

A.5 Other distributions

A.5.1.1 Lognormal—μ,σ (μ can be negative)

$$\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\
\mathrm{E}[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\
\mathrm{E}[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)] \\
\text{mode} &= \exp(\mu - \sigma^2)
\end{aligned}$$

Mid 2 Exam – Loss. 2020/S2

Solution

Ex1   3 3 5 5 6 7 8 8 8 12

(1)

y_j	s_j	r_j
3	2	10
5	2	8
6	1	6
7	1	5
8	3	4
12	1	1

$$\hat{H}_1(10) = \sum_{y_j \le 10} \frac{s_j}{r_j}$$

$$= \frac{2}{10} + \frac{2}{8} + \frac{1}{6} + \frac{1}{5} + \frac{3}{4}$$

(1) $= 1.566\ldots = \frac{47}{30}$

$3 \; 3 \; 5^+ \; 5^+ \; 6 \; 7 \; 8^+ \; 8^+ \; 8^+ \; 12$

(1)

d_j	x_j	u_j
0	3	–
	3	–
	–	5
	–	5
	6	–
	7	–
	–	8
	–	8
	–	8
0	12	

(1)

y_j	s_j	r_j
3	2	5 + 5 = 10
6	1	3 + 3 = 6
7	1	2 + 3 = 5
12	1	1

$$\hat{H}_2(10) = \frac{2}{10} + \frac{1}{6} + \frac{1}{5} = 0.566\ldots = \frac{17}{30}$$

(1)

$$\left|\hat{H}_1(10) - \hat{H}_2(10)\right| = 1.$$

Ex2

$$Y^L = \begin{cases} 0 & X \le d = 500 \\ X & X > d \end{cases}$$

$$Y^L = \max(Y-d) + d\,\mathbb{1}_{X>d}.$$

(1) $E(Y^L) = E(X) - E(X \wedge d) + d\,S(d)$

(1) $E(X) = \frac{\theta}{\alpha-1} = \frac{1000}{2} = 500$

(1) $E(X \wedge d) = \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right]$

$= 500\left[1 - \left(\frac{1000}{1500}\right)^2\right] = 277.77 = \frac{2500}{9}$

(1) $dS(d) = 500\,S(500) = 500\left(\frac{1000}{1500}\right)^3 = 148.148$ ($\left(\frac{1000}{1500}\right)^3 \to 0.2963$)

$\Rightarrow E(Y^L) = 370.3704.$

Ex 3

$X_1 \sim Exp(\text{mean} = \theta = 340)$

$f_1(x) = \frac{1}{\theta} e^{-x/\theta}$; $S(x) = e^{-x/\theta}$

$r = 30\% \Rightarrow X_2 = X_1(1+r)$

(2) $S_2(x) = P(X_2 > x) = P\left(X_1 > \frac{x}{1+r}\right) = S_1\left(\frac{x}{1+r}\right)$

$= e^{-\frac{x}{\theta(1+r)}}$

$\Rightarrow X_2 \sim Exp(\text{mean} = \theta = 340(1+0.3) = 442$

b) $d = 200.$

$E(Y^L) = E(X_2) - E(X_2 \wedge 200)$

(2) $= 442 - \underbrace{442\left(1 - e^{-200/442}\right)}_{160.869}$

$= 281.13$

or $E(Y^L) = \int_{200}^{\infty} S_2(x)\,dx = \int_{200}^{\infty} e^{-\frac{x}{442}}\,dx = 281.13$

Ex 4 $X \sim Exp(mean = \theta)$

$VX = \beta^2 = 40{,}000 \Rightarrow \theta = 200$ ①

$r = 20\%$ $\quad X_{2009} = X_{2008}(1+r)$

$$S_{X_{2009}} = P(X_{2009} > x) = P\left(X_{2008} > \frac{x}{1+r}\right)$$

$$= e^{-\frac{x}{\theta(1+r)}}.$$

① $\rightarrow X_{2009} \sim Exp(mean = \theta(1+r) = 200(1.2) = 240$

$d = 50$; $u^* = 250$.

$$Y^L = \begin{cases} 0 & X \le d \\ X - d & d < X < d + u^* = 300 \\ u^* & X > 300 = u \end{cases}$$

$$E(Y^L) = E(X \wedge u) - E(X \wedge d)$$

$$= \int_0^{300} S(x)\,dx - \int_0^{50} S(x)\,dx$$

② $$= \int_{50}^{300} S(x)\,dx = \int_{50}^{300} e^{-\frac{x}{240}}\,dx$$

$$= \left[-240\, e^{-\frac{x}{240}}\right]_{50}^{300} = 126.1018$$

OR $$E(Y^L) = 240\left(1 - e^{-\frac{300}{240}}\right) - 240\left(1 - e^{-\frac{50}{240}}\right)$$

Ex 5

d_j	x_j	u_j
0	350	–
	350	–
	500	–
	500	–
	–	500
	1000	–
	–	1000
	–	1000
	1200	–
0	1500	–

(1)

(1)

j	y_j	s_j	r_j
1	350	2	7 + 3 = 10
2	500	2	5 + 3 = 8
3	1000	1	3 + 2 = 5
4	1200	1	2 + 0 = 2
5	1500	1	1 + 0 = 1

$y_4 = 1200 \leq x = 1250 < y_5 = 1500$

$$\hat{S}(1250) = \sum_{1}^{4} \left(1 - \frac{s_i}{r_i}\right) \quad (1)$$

(1) $= (1 - 2/10)(1 - 2/8)(1 - 1/5)(1 - 1/2)$

$= 0.24 = 6/25$

Ex6

$0 < x < 35$; $f(x) = \frac{18}{1000 \times 35} = \frac{0.514}{1000}$

(4) $35 < x < 95$; $f(x) = \frac{230}{1000}\left(\frac{1}{60}\right) = \frac{3.833}{1000}$

$95 < x < 145$ $f(x) = \frac{320}{1000}\left(\frac{1}{50}\right) = \frac{6.4}{1000}$

$145 < x < 300$ $f(x) = \frac{120}{1000}\left(\frac{1}{155}\right) = \frac{0.774}{1000}$

$300 < x < 500$ $f(x) = \frac{312}{1000}\left(\frac{1}{200}\right) = \frac{1.560}{1000}$