PHYSICS 507

ASSIGNMENT-Solutions

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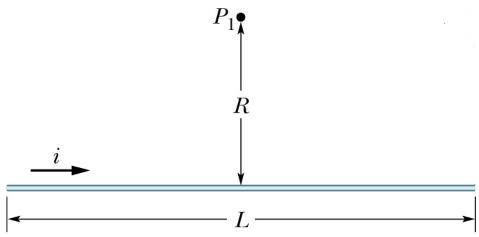
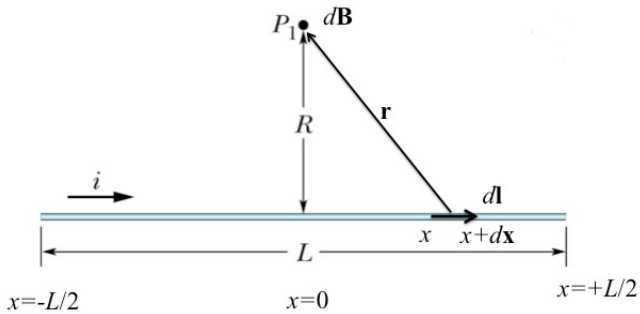
Hand in: Wednesday 28th of March 2020, time: 23:59

1. Show, that at point P_1 the magnetic field is given by: $B = \dfrac{\mu_0 i}{2\pi R}\dfrac{L}{\left(L^2+4R^2\right)^{1/2}}$. What happens if $L \to \infty$?



Solution:



To solve this problem we need to apply Biot-Savart Law. We consider the elementary part $d\mathbf{l}$ of the wire at a position x having length dx. Thus $d\mathbf{l} = dx\hat{\mathbf{x}}$. This part is flown by a current I so at the point P_1 it creates a magnetic field $d\mathbf{B}$ given by:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$$

where **r** is a vector having its tail (beginning) at the tail of d**l** and its tip (end) at the point P_1. Thus **r** = (0, h, 0) – (x, 0, 0) or **r** = (-x, h, 0). Then

$$d\mathbf{l} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ dx & 0 & 0 \\ -x & R & 0 \end{vmatrix} = Rdx\hat{\mathbf{z}}$$

The magnitude of r is given by $r = \left(x^2 + R^2\right)^{1/2}$. Thus for the elementary magnetic field we have:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{Rdx}{\left(x^2 + R^2\right)^{3/2}} \hat{\mathbf{z}}$$

The total magnetic field is taken by an integration we get

$$\mathbf{B} = \int_{-L/2}^{+L/2} d\mathbf{B} = \hat{\mathbf{z}} \frac{\mu_0 IR}{4\pi} \cdot \int_{-L/2}^{L/2} \frac{dx}{\left(x^2 + R^2\right)^{3/2}} = \hat{\mathbf{z}} \frac{\mu_0 IR}{4\pi} \cdot \left[\frac{x}{R^2\left(x^2 + R^2\right)^{1/2}} \right]_{-L/2}^{+L/2}$$

$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \cdot \frac{L}{R\left(\left(L/2\right)^2 + R^2\right)^{1/2}}$$

If L goes to infinity then

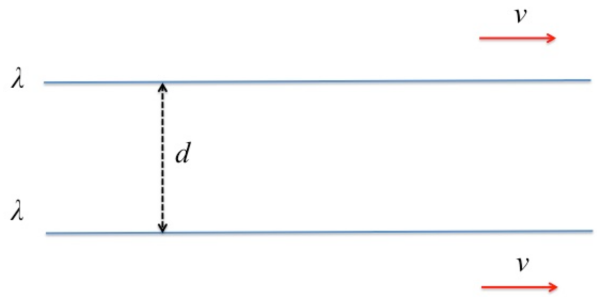
$$\mathbf{B} \approx \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi} \cdot \frac{L}{R\left(\left(L/2\right)^2\right)^{1/2}} = \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi R}$$

2. Suppose you have two infinite long straight wires carrying a charge with a linear charge density λ. The two wires move at constant speed v as shown in figure.

i) What is the magnetic force per unit length on each wire? (10 marks)

ii) What is the electric force on each wire? (6 marks)

iii) How great would v have to be in order for the magnetic attraction to balance the electric repulsion? Work out the actual number. Is it a reasonable sort of speed? (4 marks)



You are given that the magnetic field at a distance s from an infinitely current carrying wire is:

$$B = \frac{\mu_0 I}{2\pi s}$$

and the electric field at a distance s from an infinitely charged wire is

$$E = \frac{\lambda}{2\pi\varepsilon_0 s}$$

Solution

i) The moving charged wire is equivalent to a current

$$I_1 = \frac{\Delta q}{\Delta t} = \frac{\lambda\Delta x}{\Delta t} = \lambda v$$

Thus each wire generates a magnetic field at a distance d away from it:

$$B_1 = \frac{\mu_0 \lambda v}{2\pi d}$$

This field is responsible for generating a force on the other wire given by:

$$\Delta F_{2,m} = B_1 I_2 \Delta x = \frac{\mu_0 \lambda v}{2\pi d} I_2 \Delta x = \frac{\mu_0 \lambda^2 v^2}{2\pi d}\Delta x$$

Thus the magnetic force per unit length is:

$$F_m = \frac{\Delta F_{2,m}}{\Delta x} = \frac{\mu_0 \lambda^2 v^2}{2\pi d}$$

ii) The electric force per unit length on the second wire due to the electric field of the first is:

$$F_{el} = \frac{\Delta F_{2,el}}{\Delta x} = \frac{E\Delta q}{\Delta x} = \frac{\lambda}{2\pi\varepsilon_0 d\Delta x}\lambda\Delta x = \frac{\lambda^2}{2\pi\varepsilon_0 d}$$

iii) The magnetic attraction per unit length is given by:

$$F_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$$

The electric repulsion per unit length on one wire is:

$$F_{el} = \frac{1}{2\pi\epsilon_0}\frac{\lambda^2}{d}$$

The two forces balance when:

$$F_m = F_{el}$$

$$\frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d} = \frac{1}{2\pi\epsilon_0}\frac{\lambda^2}{d}$$

$$v^2 = \frac{1}{\mu_0\epsilon_0} = c^2$$

$$v = c$$

This is impossible

3. Assume you have a thin wire with current *I.* The vector potential $d\mathbf{A}$ created from an element of length $d\mathbf{l}$ of this wire at a point which is at a distance r from the element is given by the relation:

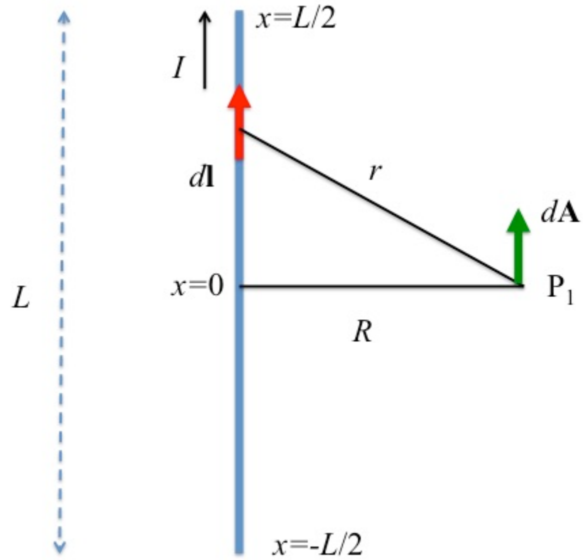
$$d\mathbf{A} = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}}{r}$$

(i) Use this relation to calculate the vector potential at point P_1 in the figure of problem 1. Assume that the distance R is far smaller than the length of the wire, i.e. $R \ll L$.

(ii) What will be the expression for the vector potential if the wire has infinite length?

You are given: $(1+x)^{1/2} \approx 1+\frac{1}{2}x$, for $x << 1$, $\int \frac{dx}{\left(a^2+x^2\right)^{1/2}} = \ln\left[x+\sqrt{x^2+a^2}\right]+C$

Solution:



We consider the elementary part $d\mathbf{l}$ of the wire at a position x having length dx. Thus $d\mathbf{l} = dx\hat{\mathbf{x}}$. This part is flown by a current I so at the point P_1 it creates a vector potential $d\mathbf{A}$ given by:

$$d\mathbf{A} = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}}{r} \Rightarrow \mathbf{A} = \left(\frac{\mu_0 I}{4\pi}\int_{-L/2}^{+L/2}\frac{dx}{r}\right)\hat{\mathbf{x}} \Rightarrow \mathbf{A} = \left(\frac{\mu_0 I}{4\pi}\int_{-L/2}^{+L/2}\frac{dx}{\left(R^2+x^2\right)^{1/2}}\right)\hat{\mathbf{x}}$$

$$\mathbf{A} = \left(\frac{\mu_0 I}{4\pi}\ln\left[x+\sqrt{x^2+R^2}\right]\Big|_{-L/2}^{L/2}\right)\hat{\mathbf{x}} = \frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{L}{2}+\sqrt{\left(\frac{L}{2}\right)^2+R^2}\right)-\ln\left(-\frac{L}{2}+\sqrt{\left(-\frac{L}{2}\right)^2+R^2}\right)\right]\hat{\mathbf{x}}$$

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{L}{2}+\sqrt{\frac{L^2}{4}+R^2}\right)-\ln\left(-\frac{L}{2}+\sqrt{\frac{L^2}{4}+R^2}\right)\right]\hat{\mathbf{x}}$$

Now since $R << L$ we have:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{L}{2}+\frac{L}{2}\sqrt{1+\frac{4R^2}{L^2}}\right)-\ln\left(-\frac{L}{2}+\frac{L}{2}\sqrt{1+\frac{4R^2}{L^2}}\right)\right]\hat{\mathbf{x}} \Rightarrow$$

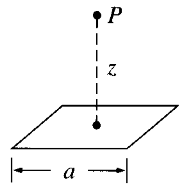
$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{L}{2}+\frac{L}{2}\left(1+\frac{2R^2}{L^2}\right)\right)-\ln\left(-\frac{L}{2}+\frac{L}{2}\left(1+\frac{2R^2}{L^2}\right)\right)\right]\hat{\mathbf{x}} \Rightarrow$$

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(L+\frac{R^2}{L}\right) - \ln\left(\frac{R^2}{L}\right)\right]\hat{\mathbf{x}} \Rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{\left(L+\frac{R^2}{L}\right)}{\left(\frac{R^2}{L}\right)}\right]\hat{\mathbf{x}}$$

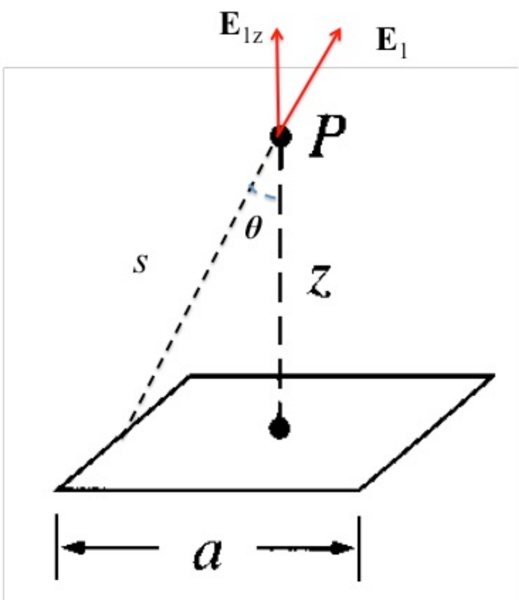
$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\ln\left(\frac{L^2}{R^2}+1\right) \underset{L>>R}{\Rightarrow} \mathbf{A} \approx \frac{\mu_0 I}{4\pi}\ln\left(\frac{L^2}{R^2}+1\right) \Rightarrow \mathbf{A} \approx \frac{\mu_0 I}{4\pi}2\ln\left(\frac{L}{R}\right)$$

$$\mathbf{A} \approx \frac{\mu_0 I}{2\pi}\ln\left(\frac{L}{R}\right)$$

4. Find the electric field (magnitude and direction) a distance z above the center of a square loop as shown in the figure, which carries a uniform line charge λ. (Hint: Use problem 2.3 above).



Solution:



From solution of problem 2.3, the electric field **from one side** on point P is given by:

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{\lambda a}{s\sqrt{s^2+a^2/4}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda a}{s\sqrt{s^2+a^2/4}}$$

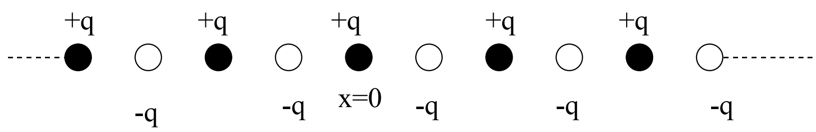
But the total field at P will be made up only from the components along the z-direction so:

$$E_p = 4E_{1z} = 4E_1 \cos\theta = 4\frac{1}{4\pi\varepsilon_0}\frac{\lambda a}{s\sqrt{s^2 + a^2/4}}\frac{z}{s}$$

$$E_p = \frac{1}{\pi\varepsilon_0}\frac{\lambda a z}{\left(z^2 + a^2/4\right)\sqrt{z^2 + a^2/2}}$$

5. On a straight line we place alternatively an infinite number of charges $+q$ and $-q$ at equal distance as shown in the figure. What is the potential energy of a charge $+q$? You are given that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \left(-1\right)^{n+1}\frac{x^n}{n} + \ldots \qquad \left(-1 < x \le 1\right)$$



Solution:

Let the distance between two adjacent charges be *d.* The potential energy of the charge $+q$ at $x = 0$ is given by

$$U = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{d} - \frac{q^2}{d} + \frac{q^2}{2d} + \frac{q^2}{2d} - \frac{q^2}{3d} - \frac{q^2}{3d}\right) = \frac{1}{4\pi\varepsilon_0}\left(-\frac{2q^2}{d} + \frac{2q^2}{2d} - \frac{2q^2}{3d} + \ldots\right)$$

$$= -\frac{2q^2}{4d\pi\varepsilon_0}\left(1 - \frac{1}{2} + \frac{1}{3} + \ldots\right) = -\frac{q^2}{2d\pi\varepsilon_0}\left(1 - \frac{1^2}{2} + \frac{1^3}{3} + \ldots\right) = -\frac{q^2}{2d\pi\varepsilon_0}\ln(1+1) = -\frac{q^2}{2d\pi\varepsilon_0}\ln 2$$