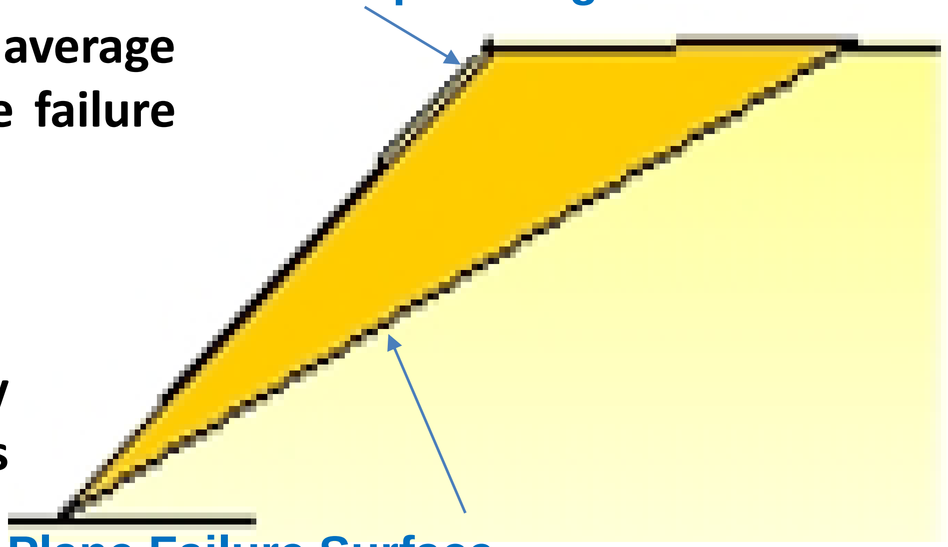
# Stability of Finite Slopes with Plane Failure Surface

- **For simplicity, when analyzing the stability of a finite slope in a homogeneous soil, we need to make an assumption about the general shape of the surface of potential failure.**
- **The simplest approach is to approximate the surface of potential failure as a plane.**
- **However, considerable evidence suggests that slope failures usually occur on curved failure surfaces**
- **Hence most conventional stability analyses of slopes have been made by assuming that the curve of potential sliding is an arc of a circle.**

# Culmann's Method

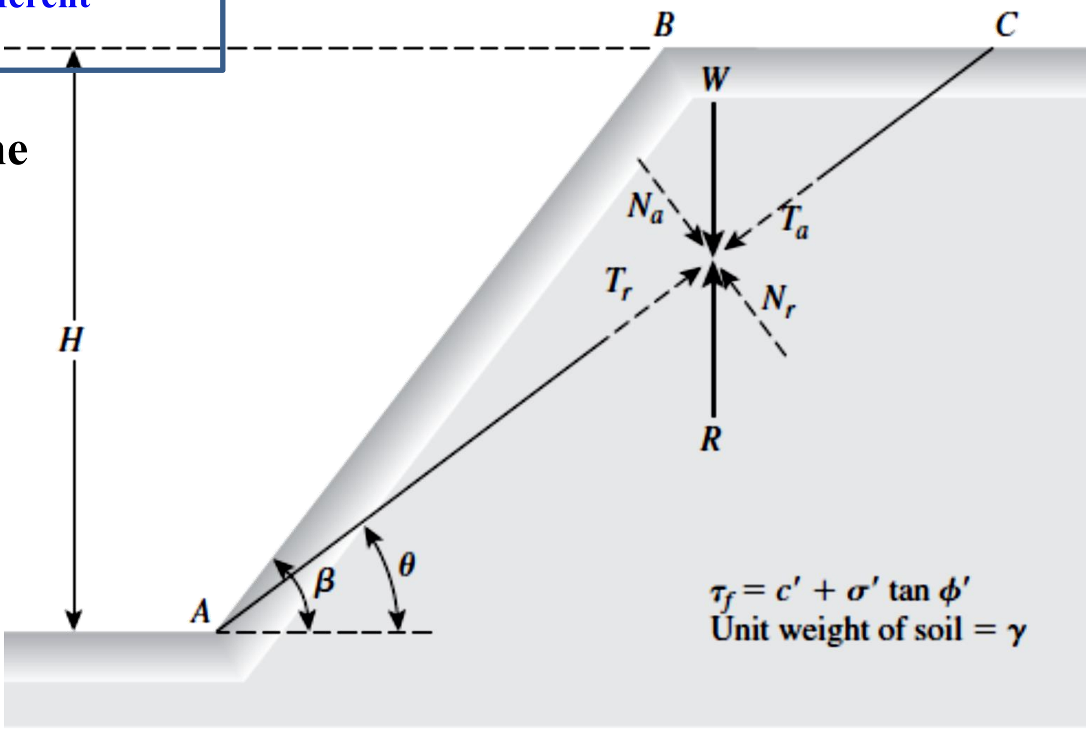
- Culmann's method assumes that the critical surface of failure is a plane surface passing through the toe.
- Culmann's analysis is based on the assumption that the failure of a slope occurs along a plane when the average shearing stress tending to cause the slip is more than the shear strength of the soil.
- Also, the most critical plane is the one that has a minimum ratio of the average shearing stress that tends to cause failure to the shear strength of soil.
- The method gives reasonably accurate results if the slope is vertical or nearly vertical.



# Culmann's Method

- A slope of height **H** and that rises at an angle **β** is shown below.
- The forces that act on the mass are shown in the figure, where trial failure plane ***AC is*** inclined at angle **θ** with the horizontal.

Similar procedures as for infinite slope, only different geometry. Also here we made optimization.




**The average shear stress on the plane *AC***

$$\tau = \frac{T_a}{(\overline{AC})(1)} = \frac{T_a}{\left(\frac{H}{\sin\theta}\right)}$$

$T_a = W \sin\theta$

$$W = \frac{1}{2}(H)(\overline{BC})(1)(\gamma) = \frac{1}{2}H(H\cot\theta - H\cot\beta)\gamma$$

$$= \frac{1}{2}\gamma H^2\left[\frac{\sin(\beta-\theta)}{\sin\beta\,\sin\theta}\right]$$

$$\tau = \frac{1}{2}\gamma H\left[\frac{\sin(\beta-\theta)}{\sin\beta\,\sin\theta}\right]\sin^2\theta \quad (*)$$

# Culmann's Method

The average resistive shearing stress (Developed shear strength) developed along the plane *AC* also may be expressed as

$$\tau_d = c'_d + \sigma' \tan\phi'_d$$

$$\sigma' = \frac{N_a}{(\overline{AC})(1)} = \frac{N_a}{\left(\dfrac{H}{\sin\theta}\right)}$$

$$N_a = \text{normal component} = W\cos\theta$$

$$N_a = \frac{1}{2}\gamma H^2\left[\frac{\sin(\beta-\theta)}{\sin\beta\ \sin\theta}\right]\cos\theta$$

$$\sigma' = \frac{1}{2}\gamma H\left[\frac{\sin(\beta-\theta)}{\sin\beta\ \sin\theta}\right]\cos\theta\ \sin\theta$$

$$\tau_d = c'_d + \frac{1}{2}\gamma H\left[\frac{\sin(\beta-\theta)}{\sin\beta\ \sin\theta}\right]\cos\theta\ \sin\theta\ \tan\phi'_d \quad (**)$$

**Equating the R.H.S of Eqs. (*) and (**) gives**

$$\boxed{c'_d = \frac{1}{2}\gamma H\left[\frac{\sin(\beta-\theta)(\sin\theta - \cos\theta\tan\phi'_d)}{\sin\beta}\right]} \quad (***)$$

# Culmann's Method

## Critical failure plane

- **The expression in Eq. (***) is derived for the trial failure plane AC.**

- **To determine the critical failure plane, we must use the principle of maxima and minima (for $F_s$=1 and for given values of c', ϕ', g, H, β) to find the critical angle θ:**

$$\frac{\partial c'_d}{\partial \theta} = 0 \Rightarrow \boxed{\theta_{cr} = \frac{\beta + \phi'}{2}}$$

- **Substitution of the value of $\theta = \theta_{cr}$ into Eq. (***) yields**

$$\boxed{c'_d = \frac{\gamma H}{4}\left[\frac{1 - \cos(\beta - \phi'_d)}{\sin\beta \cos\phi'_d}\right]} \qquad (****)$$

# Culmann's Method

$$\boxed{\frac{c'_d}{\gamma H} = m = \frac{1 - \cos(\beta - \phi'_d)}{4 \sin\beta \cos\phi'_d}}$$

The **maximum height** of the slope for which critical equilibrium occurs can be obtained by substituting $c'_d = c'$ and $\phi'_d = \phi'$ (****)

$$\boxed{H_{cr} = \frac{4c'}{\gamma}\left[\frac{\sin\beta\cos\phi'}{1 - \cos(\beta - \phi')}\right]}$$

- **For purely cohesive soils c ≠ 0 ϕ = 0.**

$$\theta_{cr} = \frac{\beta}{2} \longrightarrow \boxed{H_{cr} = \frac{4c'}{\gamma}\left[\frac{\sin\beta}{1 - \cos\beta}\right]}$$

# Culmann's Method

- **Steps for Solution**

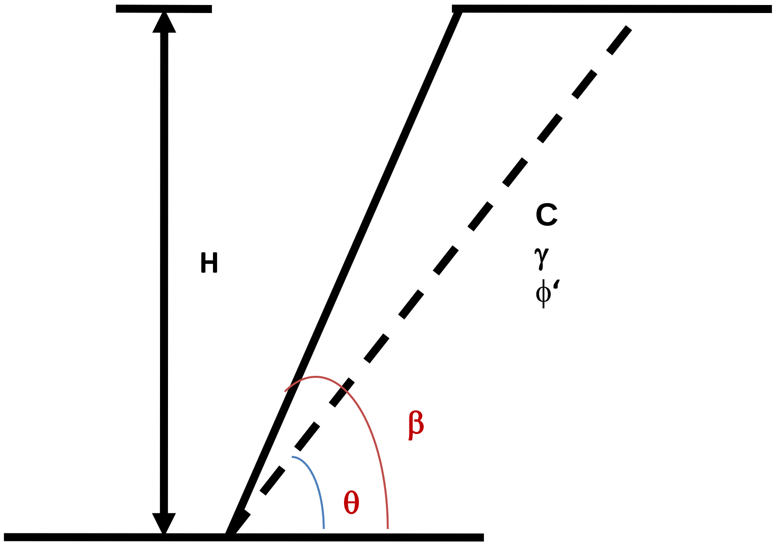
## A. If $F_s$ is given; H is required

1. $F_c = F_\phi = F_s$

2. $c_d' = \dfrac{c'}{F_s}$

3. $\tan\phi_d' = \dfrac{\tan\phi'}{F_s}$

4. $H = \dfrac{4c_d'}{\gamma}\left(\dfrac{\sin\beta\cos\phi_d'}{1-\cos(\beta-\phi_d')}\right)$

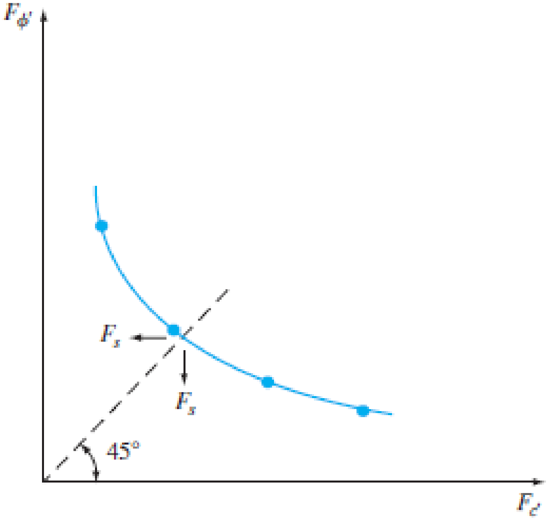


# Culmann's Method

- **Steps for Solution**

## B. If H is given; $F_s$ is required

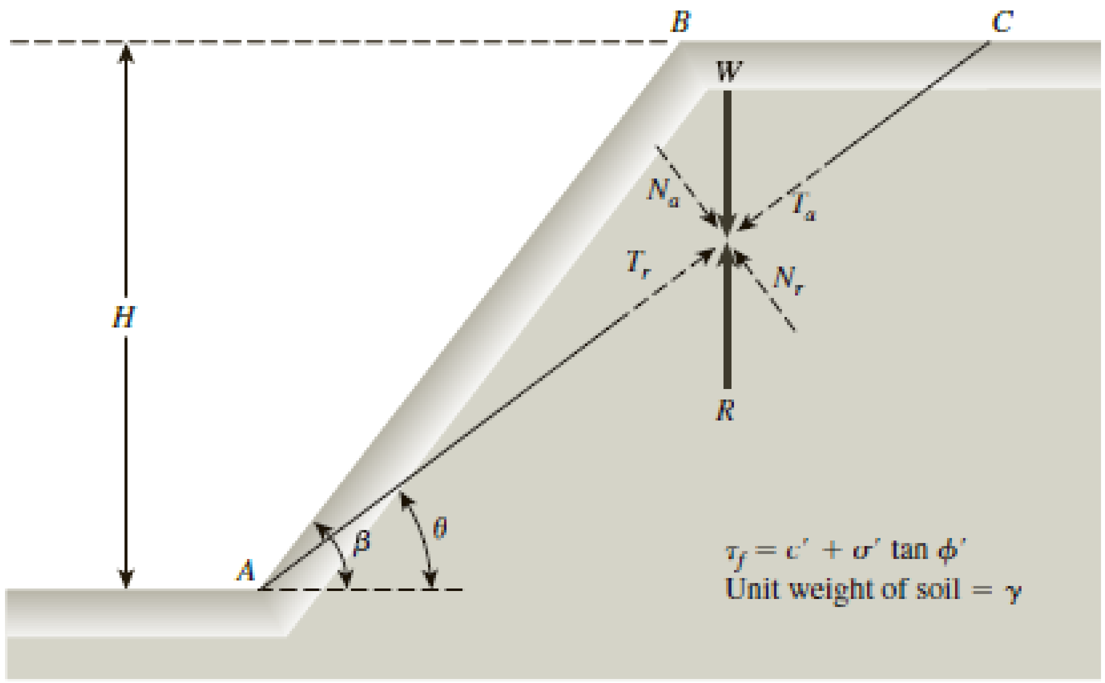
1. Assume $F_\phi$
2. $\tan\phi_d' = \dfrac{\tan\phi'}{F_s}$
3. $c_d' = \dfrac{\gamma H}{4}\left(\dfrac{1-\cos(\beta-\phi_d')}{\sin\beta\cos\phi_d'}\right)$
4. $F_c = \dfrac{c'}{c_d'}$
5. Check if $F_c = F_\phi \rightarrow F_s = F_c = F_\phi$
6. If $F_c \neq F_\phi \rightarrow$ try another $F_\phi$
7. Repeat steps $1 \rightarrow 5$



# EXAMPLE 15.3

## Example 15.3

Refer to Figure 15.9. For a trial failure surface $AC$ in the slope, given: $H = 5$ m, $\beta = 55°$, $\theta = 35°$, and $\gamma = 17.5$ kN/m$^3$. The shear strength parameters of the soil are $c' = 25$ kN/m$^2$ and $\phi' = 26°$. Determine the factor of safety $F_s$ for the trial failure surface.



# EXAMPLE 15.3

**Solution**

From Eq. (15.33), the average shear stress on the plane $AC$ is

$$\tau = \frac{1}{2}\gamma H\left[\frac{\sin(\beta-\theta)}{\sin\beta\cdot\sin\theta}\right]\sin^2\theta$$

$$= \left(\frac{1}{2}\right)(17.5)(5)\left[\frac{\sin(55-35)}{(\sin 55)(\sin 35)}\right]\sin^2 35 = 10.48\ \text{kN/m}^2$$

From Eq. (15.34), the maximum average shear stress that can be mobilized on the plane $AC$ is

$$\tau_f = c' + \frac{1}{2}\gamma H\left[\frac{\sin(\beta-\theta)}{\sin\beta\sin\theta}\right]\cos\theta\sin\theta\tan\phi'$$

$$= 25 + \left(\frac{1}{2}\right)(17.5)(5)\left[\frac{\sin(55-35)}{(\sin 55)(\sin 35)}\right](\cos 35)(\sin 35)(\tan 26)$$

$$= 32.3\ \text{kN/m}^2$$

So,

$$F_s = \frac{32.3}{10.48} = \mathbf{3.08}$$

# EXAMPLE

**A cut is to be made in a soil having properties as shown in the figure below.**

**If the failure surface is assumed to be finite plane, determine the followings:**

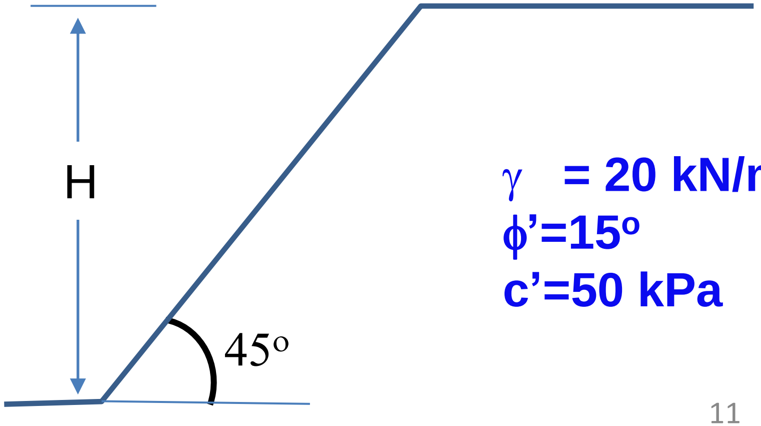
**(a) The angle of the critical failure plane.**

**(b) The critical depth of the cut slope**

**(c) The safe (design) depth of the cut slope. Assume the factor of safety ($F_s$=3)?**

**Given equation:**

$$H_{cr} = \frac{4c'}{\gamma}\left[\frac{\sin\beta\cos\phi'}{1-\cos(\beta-\phi')}\right]$$




$\gamma$ = 20 kN/m$^3$
$\phi'$=15°
c'=50 kPa

# Key Solution

**(a)** The angle of the critical failure plane $\theta$ can be calculated from:

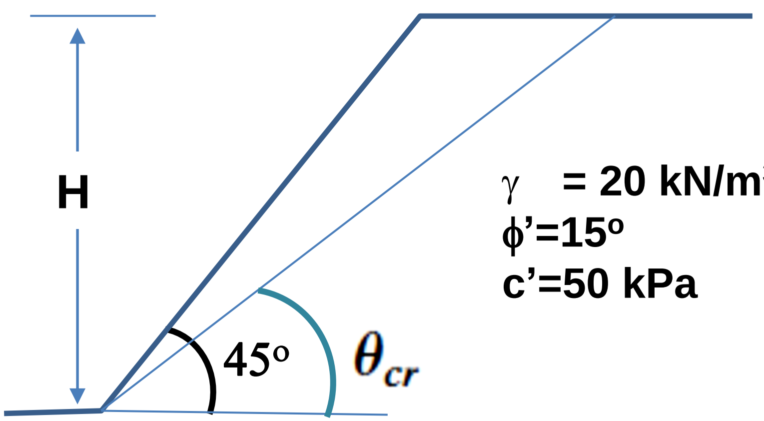
$$\theta_{cr} = \frac{\beta + \phi'}{2}$$

$\beta = 45^o$

$\phi' = 15^o$

**(b)** The critical depth of the cut slope can be calculated from:

$$H_{cr} = \frac{4c'}{\gamma}\left[\frac{\sin\beta\cos\phi'}{1-\cos(\beta-\phi')}\right]$$




$\gamma$ = 20 kN/m$^3$

$\phi'$=15$^o$

c'=50 kPa

**(c)** The safe (design) depth of the cut slope.

$$H_d = \frac{4c'_d}{\gamma}\left[\frac{\sin\beta\cdot\cos\phi'_d}{1-\cos(\beta-\phi'_d)}\right]$$

where: $c'_d$ and $\phi'_d$ can be determined from:

$$c'_d = \frac{c'}{F_{c'}} = \frac{c'}{F_s}$$

$$\tan\phi'_d = \frac{\tan\phi'}{F_{\phi'}} = \frac{\tan\phi'}{F_s}$$

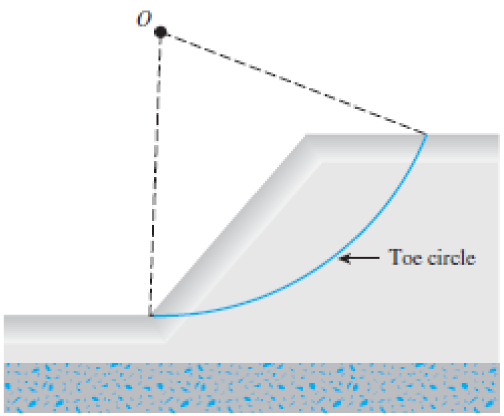
# Finite Slopes with Circular Failure Surface

## Modes of Failure

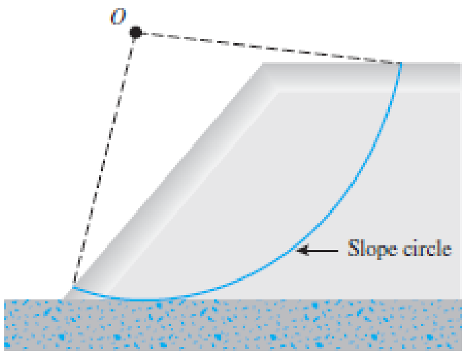
### i. Slope failure

- Surface of sliding intersects the slope at or above its toe.

  1. The failure circle is referred to as a *toe circle* if it passes through the toe of the slope

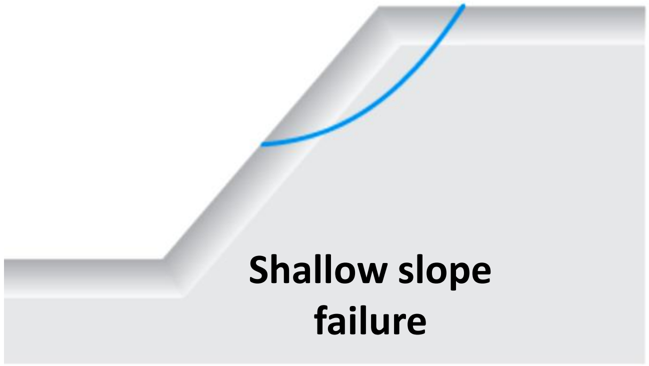



  1. The failure circle is referred to as a *slope circle* if it passes above the toe of the slope.




### ii. Shallow failure

Under certain circumstances, a *shallow slope failure* can occur.

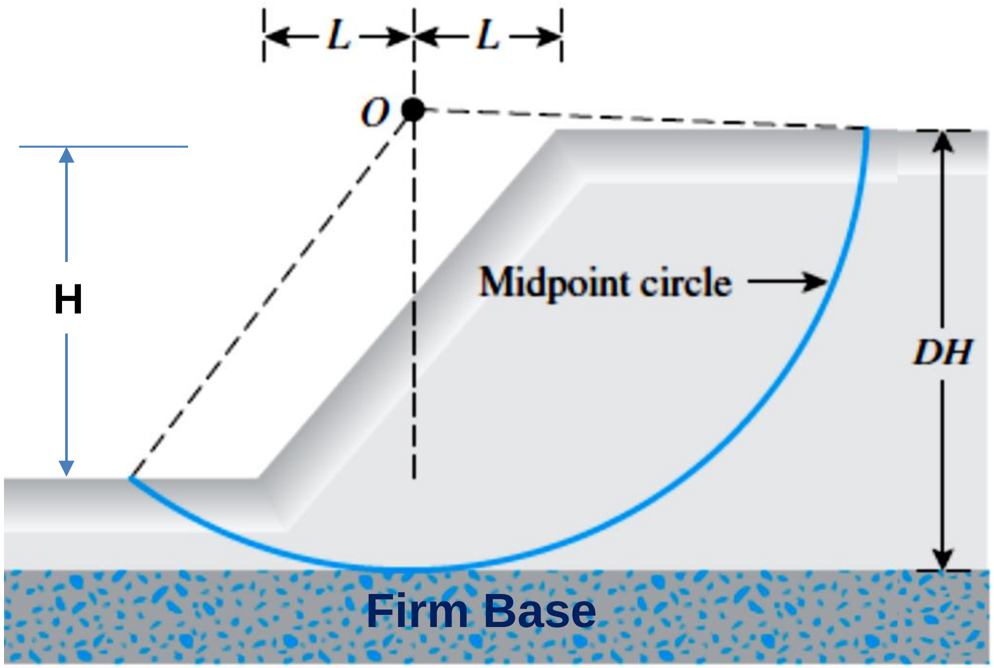


# Finite Slopes with Circular Failure Surface

## iii. Base failure

- The surface of sliding passes at some distance below the toe of the slope.

- The circle is called the midpoint circle because its center lies on a vertical line drawn through the midpoint of the slope.




- For $\beta > 53^o$ always toe

- For $\beta < 53^o$ could be toe, slope, or midpoint and that depends on depth function D where:

**Depth function:**

$$D = \frac{\text{Vertical distance from top of slope to firm base}}{\text{Height of slope}}$$

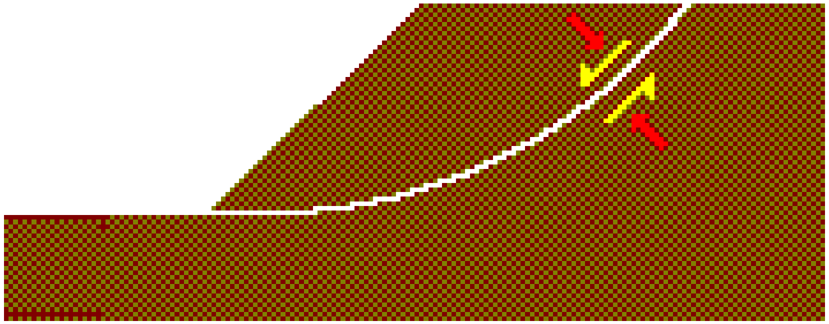
# Finite Slopes with Circular Failure Surface

- **Summary**
- **Toe Circle** — **all circles for soils with $\phi > 3°$ & $\beta > 53°$**
- **Slope Circle** — **always for $D = 0$ & $\beta < 53°$**
- **Midpoint Circle** — **always for $D > 4$ & $\beta < 53°$**

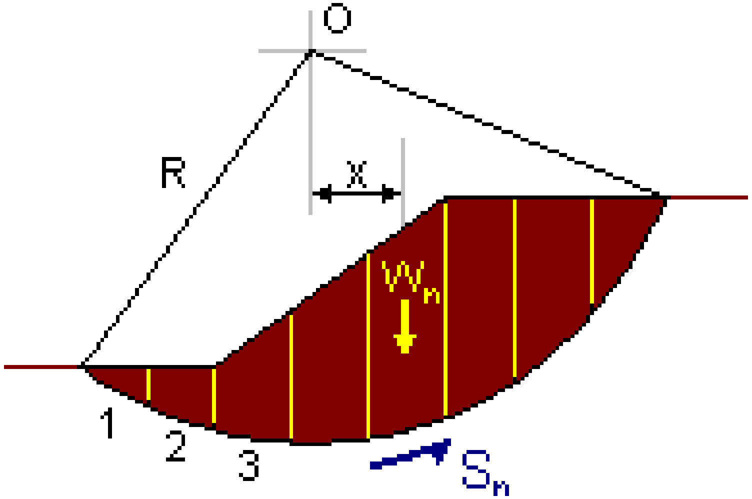
# Types of Stability Analysis Procedures

Various procedures of stability analysis may, in general, be divided into two major classes:

1. Mass procedure
   - In this case, the mass of the soil above the surface of sliding is taken as a unit.
   - This procedure is useful when the soil that forms the slope is assumed to be homogeneous.



2. Method of slices
   - Most natural slopes and many man-made slopes consist of more than on soil with different properties.
   - In this case the use of mass procedure is inappropriate.



# Mass Procedure

**1.Slopes in purely cohesionless soil with c = 0, $\phi \neq 0$**

**Failure generally does not take place in the form of a circle. So we will not go into this analysis.**

**2. Slopes in Homogeneous clay Soil with $c \neq 0$ , $\phi = 0$**

**Determining factor of safety using equilibrium equations (Case I)**

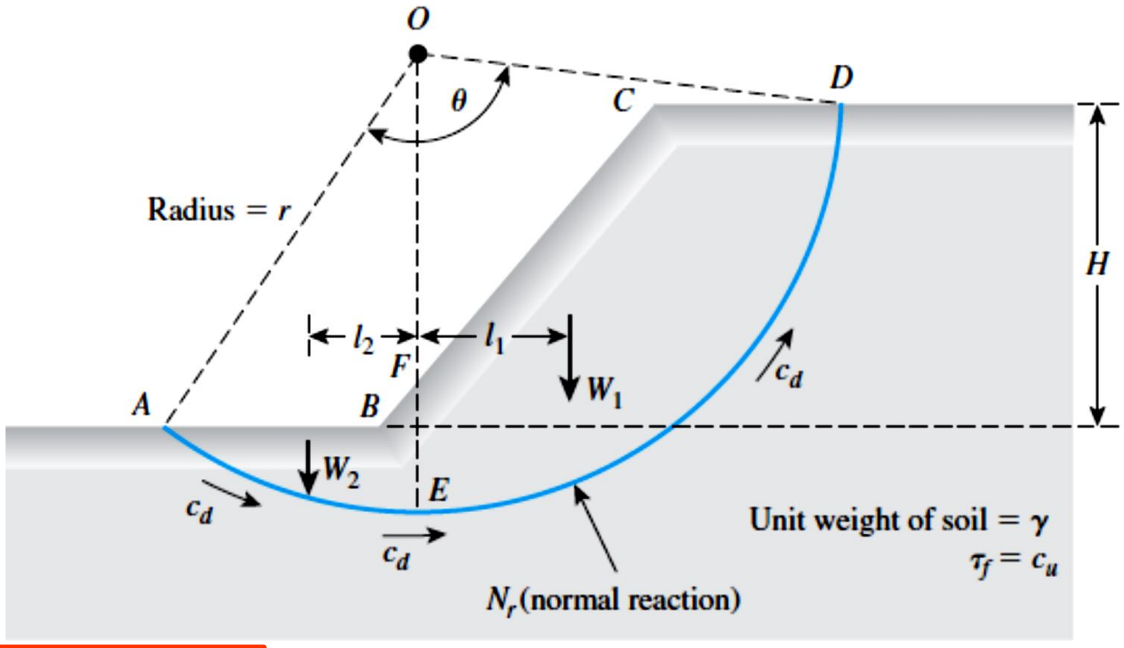
$M_{driving} = M_d = W_1 l_1 - W_2 l_2$

$W_1$ = (area of FCDEF) $\gamma$

$W_2$ = (area of ABFEA) $\gamma$

$M_{resisting} = M_R = c_d$ (AED) (1) r

$= c_d r^2 \theta$




For equilibrium, $M_R = M_d$; thus,

$$c_d = \frac{W_1 l_1 - W_2 l_2}{r^2 \theta}$$

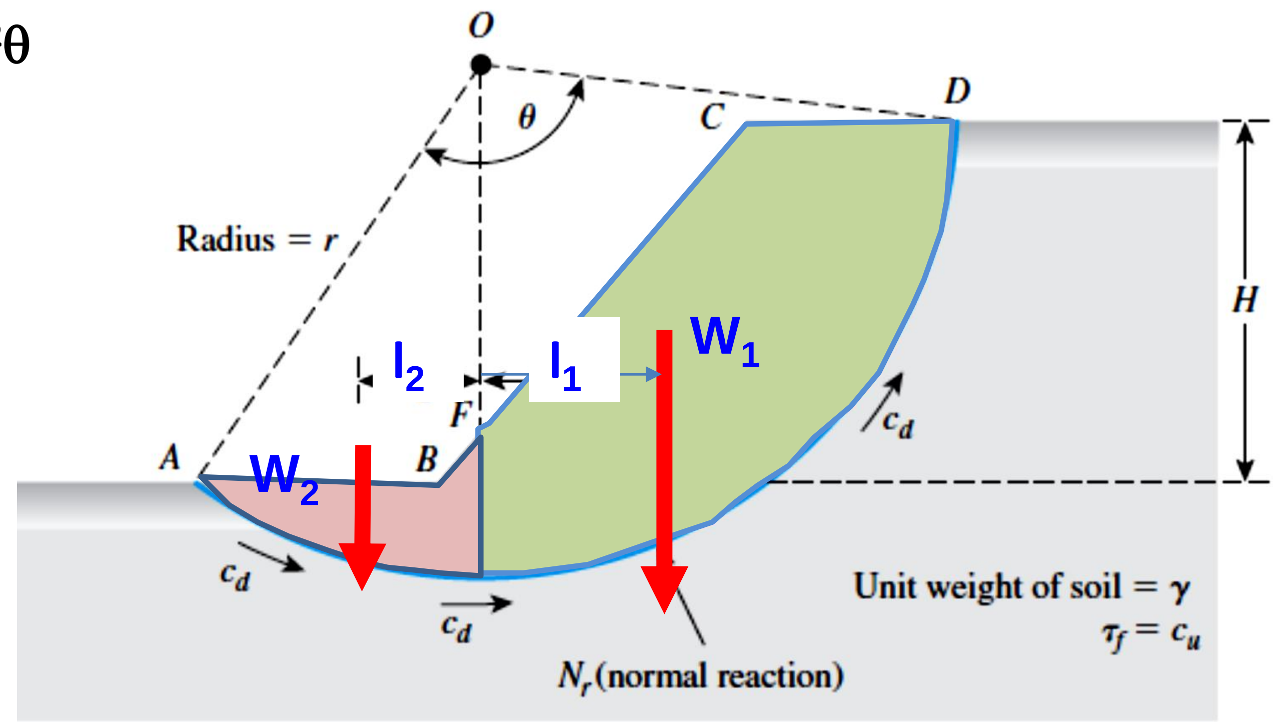
$$F_s = \frac{\tau_f}{c_d} = \frac{c_u}{c_d}$$

$$\boxed{F_s = \frac{r^2 \theta c_u}{w_1 l_1 - w_2 l_2}}$$

# Mass Procedure

$M_{driving} = M_d = W_1 l_1 - W_2 l_2$

$M_{resisting} = M_R = c_d\, r^2 \theta$




$$F.S. = \frac{M_{resisting}}{M_{driving}}$$

$$F_s = \frac{\tau_f}{c_d} = \frac{c_u}{c_d}$$

$$F_s = \frac{r^2 \theta c_u}{w_1 l_1 - w_2 l_2}$$

# REMARKS

- The potential curve of sliding, *AED,* was chosen arbitrarily.
- The critical surface is that for which the ratio of $C_u$ to $C_d$ is a minimum. In other words, $C_d$ is maximum.
- To find the critical surface for sliding, one must make a number of trials for different trial circles.
- The minimum value of the factor of safety thus obtained is the factor of safety against sliding for the slope, and the corresponding circle is the critical circle.

# Finite Slopes with Circular Failure Surface

- **Fellenius (1927) and Taylor (1937) have analytically solved for the *minimum factor of safety* and *critical circles.***

- **They expressed the developed cohesion as**

$$c_d = \gamma\ H\ m$$

Where

m = Stability number

H = height of slope

γ = unit weight of soil

$$\text{or} \quad m = \frac{c_d}{\gamma\ \text{H}}$$

- **We then can calculate the min $F_s$ as**

$$F_s = \frac{\tau_f}{c_d} = \frac{c_u}{c_d}$$

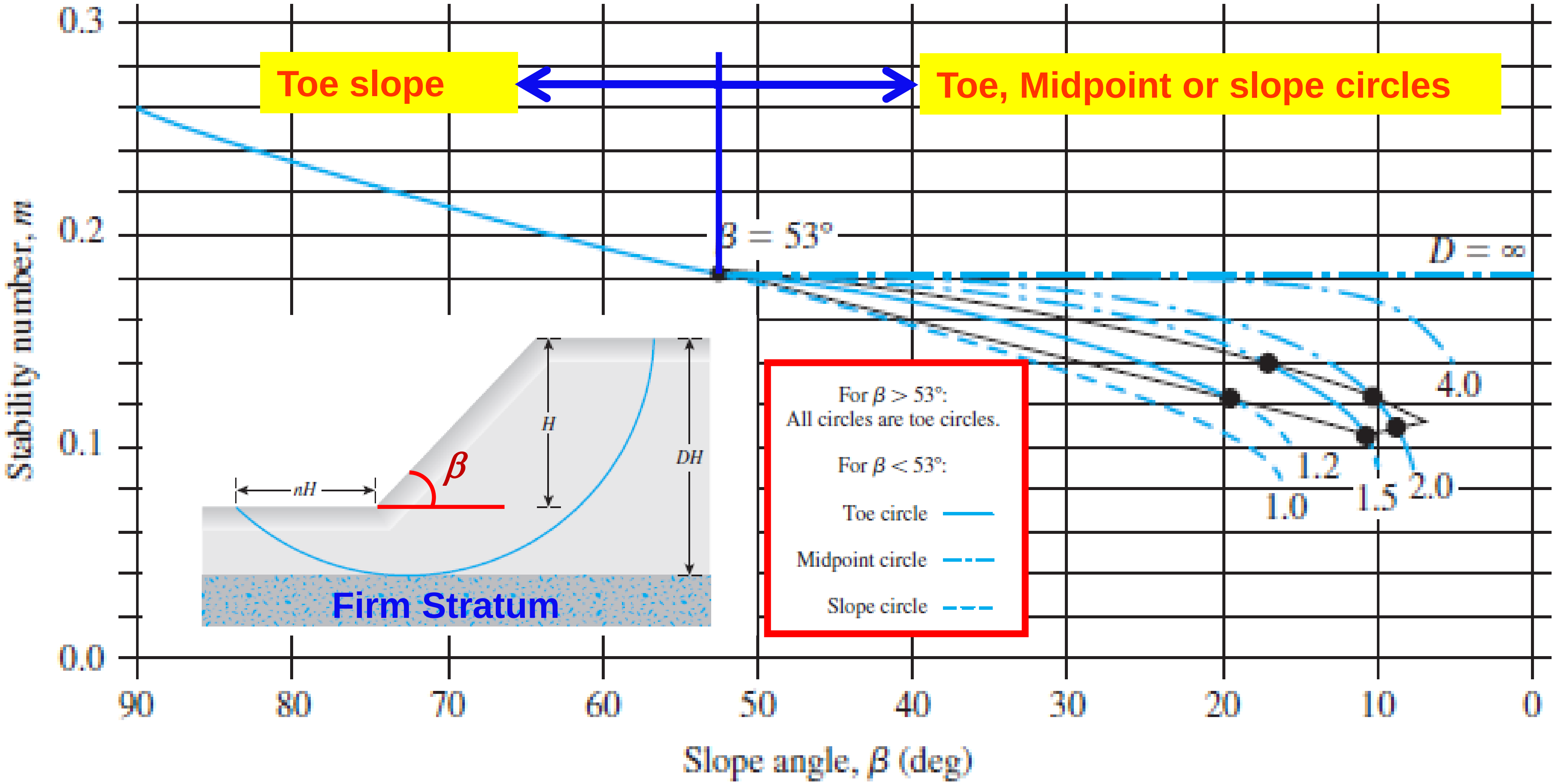
$$F_s = \frac{c_u}{\gamma H m}$$

- **The critical height (i.e., *Fs* 1) of the slope can be evaluated by substituting $H = H_{cr}$ and $c_d = c_u$ (full mobilization of the undrained shear strength) into the preceding equation. Thus,**

$$H_{cr} = \frac{c_u}{\gamma m}$$

# Finite Slopes with Circular Failure Surface

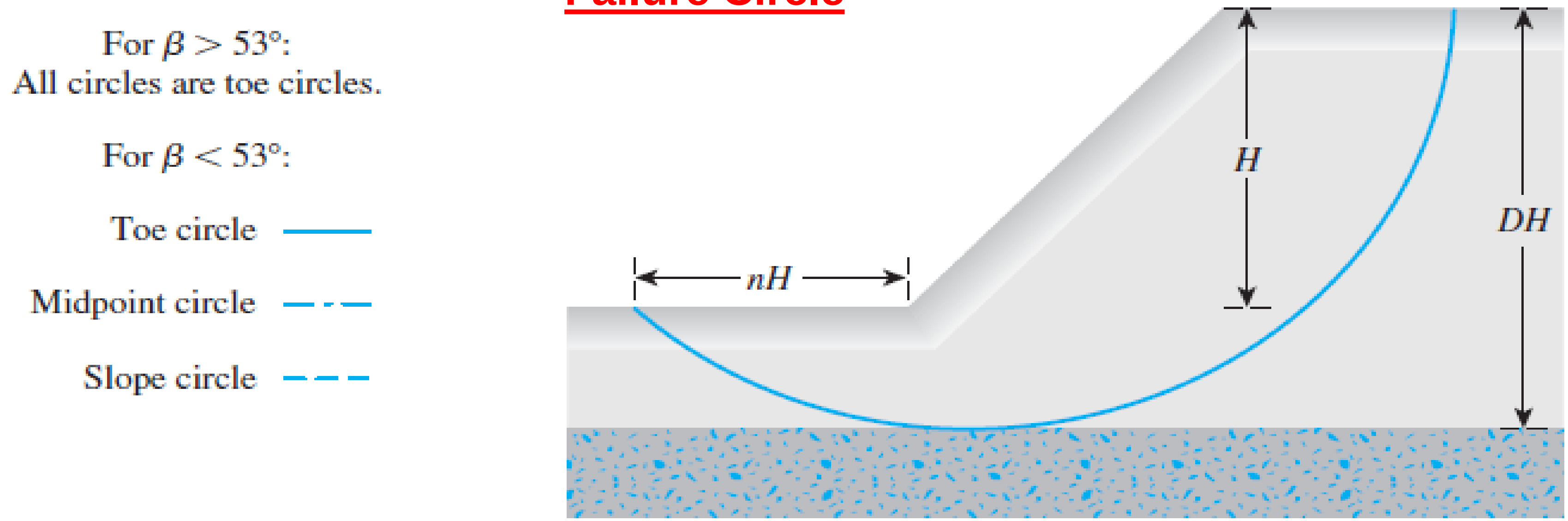
- The results of analytical solution to obtain critical circles was represented graphically as the variation of stability number, $m$ , with slope angle $\beta$.




**m is obtained from this chart depending on angle β**

# Finite Slopes with Circular Failure Surface

**Failure Circle**




- **For a slope angle $\beta > 53°$, the critical circle is always a toe circle. The location of the center of the critical toe circle may be found with the aid of Figure 15.13.**

- **For β < 53°, the critical circle may be a toe, slope, or midpoint circle, depending on the location of the firm base under the slope. This is called the depth function, which is defined as**

$$D = \frac{\text{Vertical distance from top of slope to firm base}}{\text{Height of slope}}$$

# Location of the center of the critical toe circle

For $\beta > 53°$:
All circles are toe circles.

The location of the center of the critical toe circle may be found with the aid of Figure 15.13

(radius) $$r = \frac{H_{cr}}{2 \sin \alpha \quad \sin \frac{\theta}{2}}$$

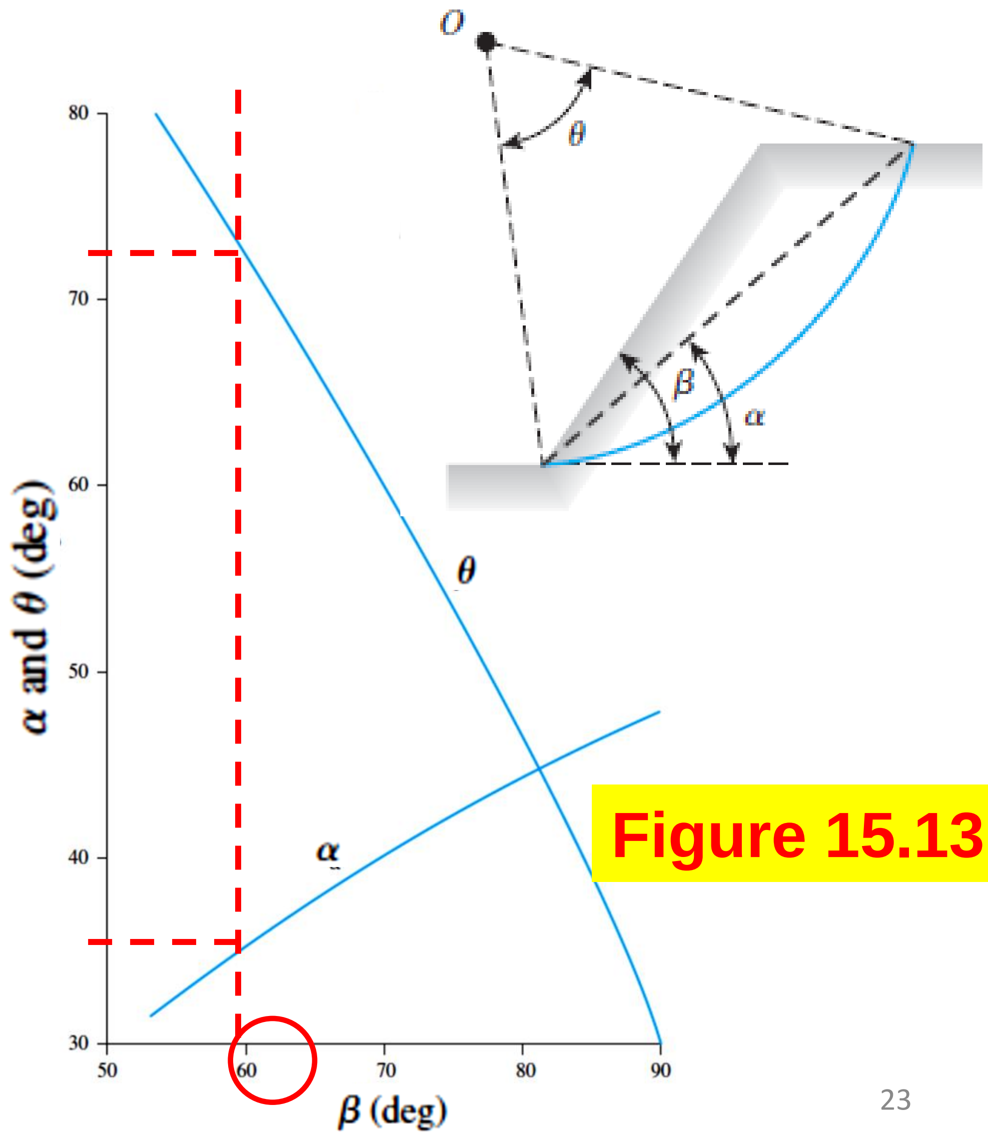



Figure 15.13

# Finite Slopes with Circular Failure Surface

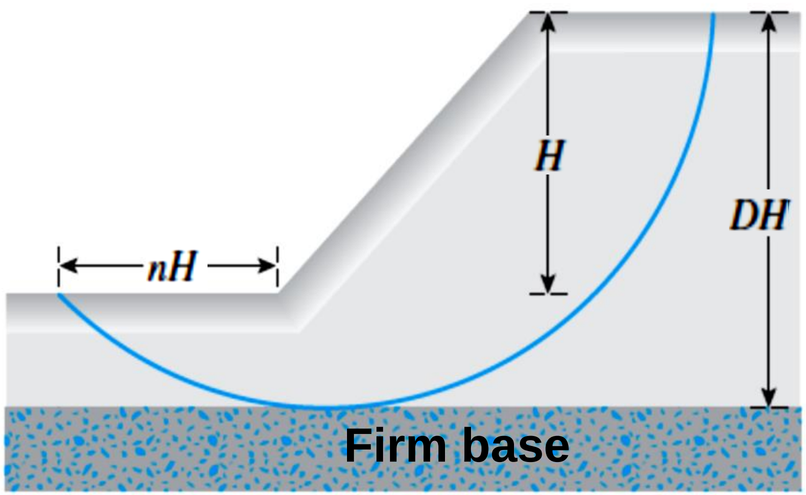
2. For $\beta < 53°$, the critical circle may be a toe, slope, or midpoint circle, depending on the location of the firm base under the slope. This is called the *depth function*, which is defined as

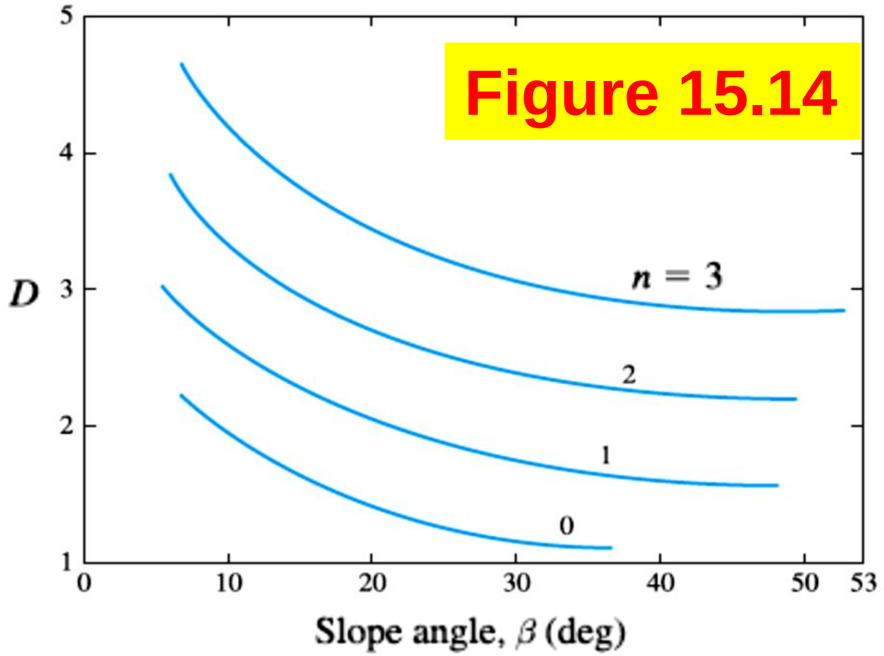
$$D = \frac{\text{Vertical distance from top of slope to firm base}}{\text{Height of slope}} \quad (15.49)$$

For $\beta < 53°$:

midpoint circle

**When the critical circle is a midpoint circle (i.e., the failure surface is tangent to the firm base), its position can be determined with the aid of Figure 15.14.**



## Critical toe circles for slopes with $\beta < 53°$

For $\beta < 53°$:

toe circles

**Table 15.1** Location of the Center of Critical Toe Circles ($\beta < 53°$)

| $n'$ | $\beta$ (deg) | $\alpha_1$ (deg) | $\alpha_2$ (deg) |
|---|---|---|---|
| 1.0 | 45 | 28 | 37 |
| 1.5 | 33.68 | 26 | 35 |
| 2.0 | 26.57 | 25 | 35 |
| 3.0 | 18.43 | 25 | 35 |
| 5.0 | 11.32 | 25 | 37 |

*Note*: for notations of $n'$, $\beta$, $\alpha_1$, and $\alpha_2$, see Figure 15.16

**The location of these circles can be determined with the use of Figure 15.15 and Table 15.1.**

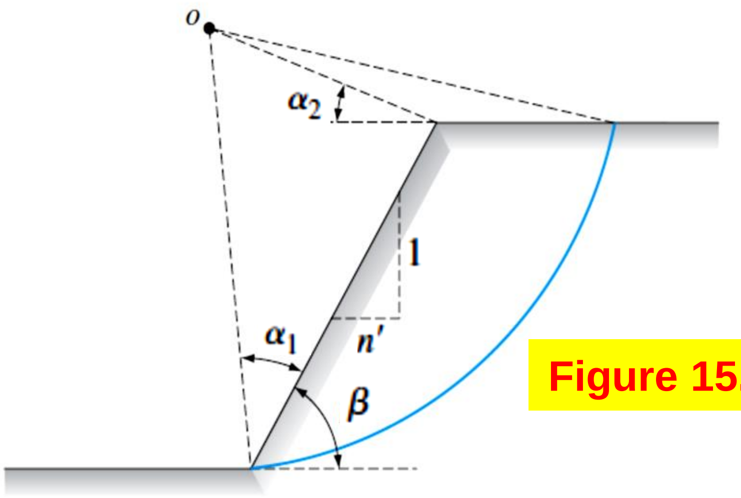
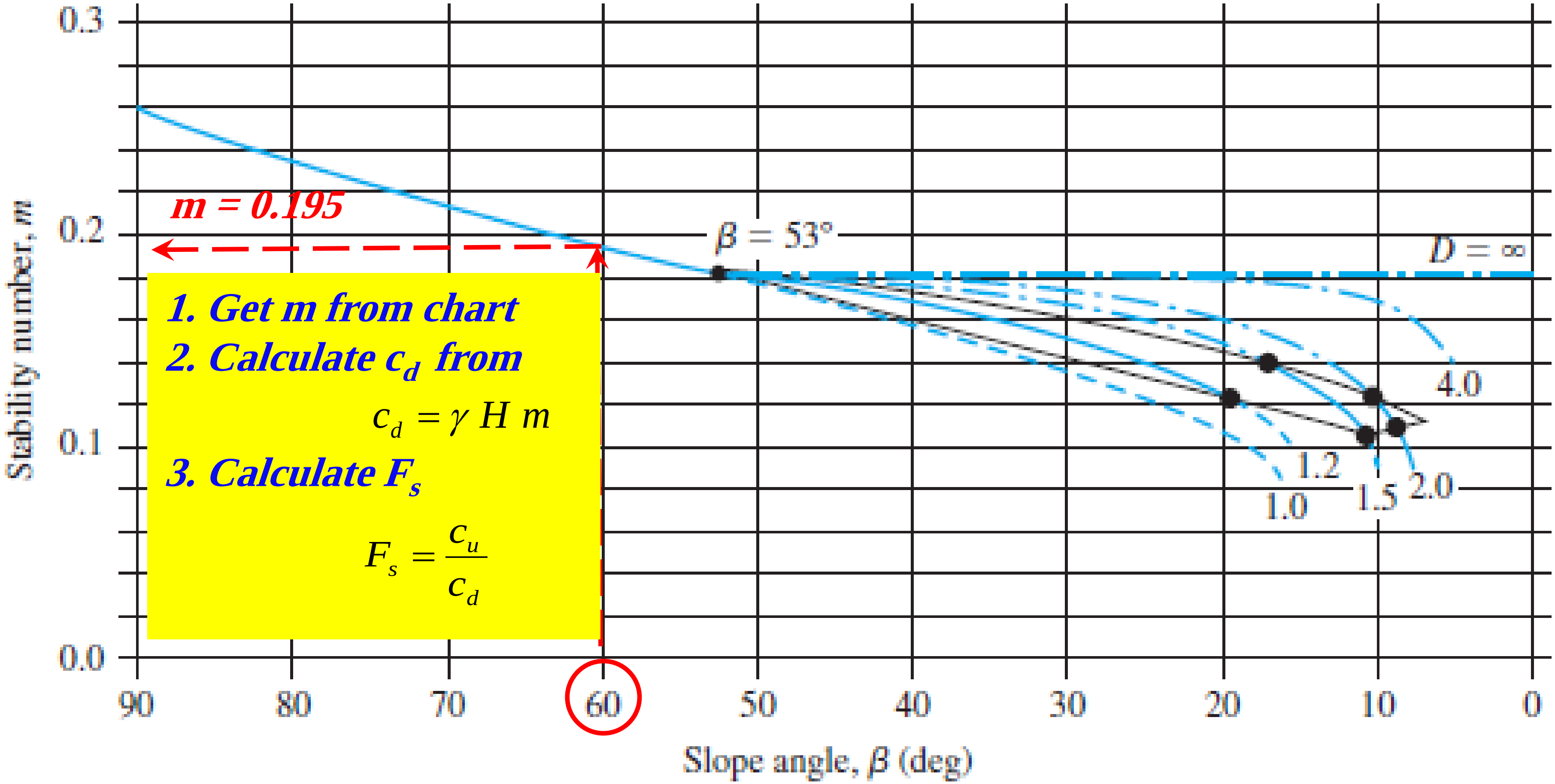



Figure 15.15

**Figure 15.16** Location of the center of critical toe circles for $\beta < 53°$

**Note that these critical toe circle are not necessarily the most critical circles that exist.**

## How to use the stability chart? Given: $\beta = 60^o$, H, $\gamma$, $c_u$  Required: min $F_s$




*1. Get m from chart*

*2. Calculate $c_d$ from*

$$c_d = \gamma\, H\, m$$

*3. Calculate $F_s$*

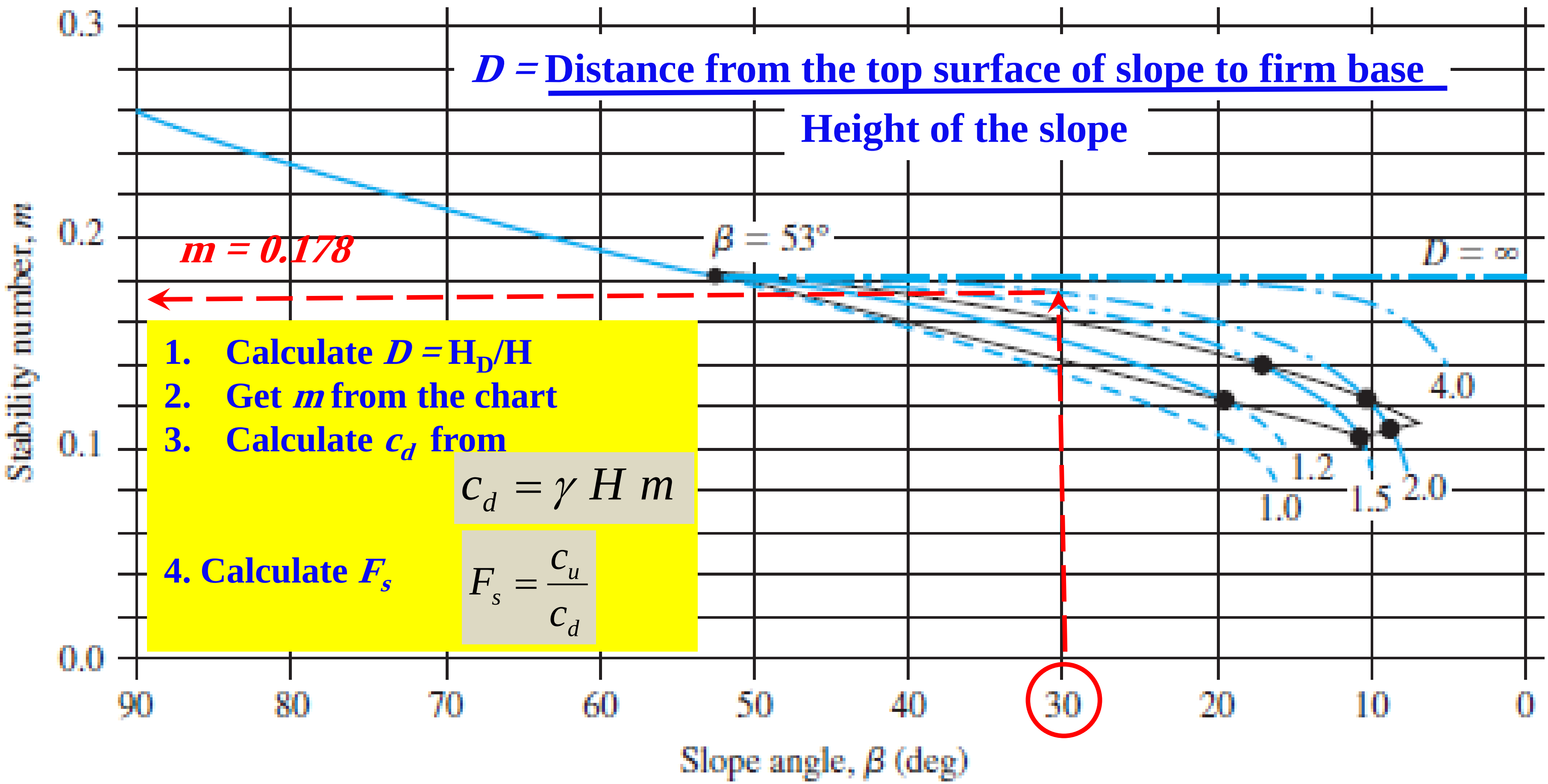
$$F_s = \frac{c_u}{c_d}$$

should be careful in using Figure 15.13 and note that it is valid for slopes of saturated clay and is applicable to only undrained conditions ($\phi = 0$).

Terzaghi used the term $\gamma H/c_d$, the reciprocal of $m$ and called it the *stability factor.*

# How to use the previous chart?

Given: $\beta = 30^o$, $H$, $\gamma$, $c_u$, $H_D$ (depth to hard stratum)  Required: min. $F_s$




Note that recent investigation put angle β at 58º instead of the 53º value.

# EXAMPLE 15.4

## Example 15.4

A cut slope in saturated clay (Figure 15.16) makes an angle 56° with the horizontal.

a. Determine the maximum depth up to which the cut could be made. Assume that the critical surface for sliding is circularly cylindrical. What will be the nature of the critical circle (i.e., toe, slope, or midpoint)?
b. How deep should the cut be made if a factor of safety of 2 against sliding is required?

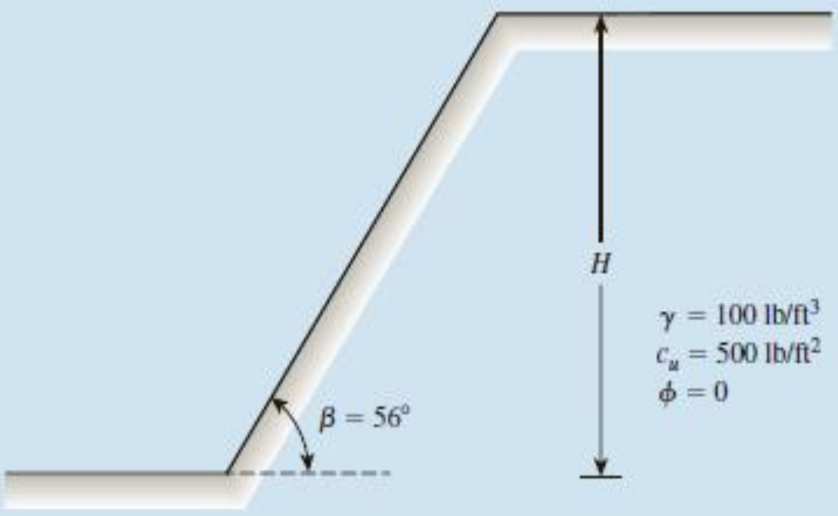



Figure 15.16

# EXAMPLE 15.4

## Solution

**Part a**

Since the slope angle $\beta = 56° > 53°$, the critical circle is a **toe circle**. From Figure 15.12 for $\beta = 56°$, $m = 0.185$. Using Eq. (15.48), we have

$$H_{cr} = \frac{c_u}{\gamma m} = \frac{500}{(100)(0.185)} = \mathbf{27.03\ ft}$$

**Part b**

The developed cohesion is

$$c_d = \frac{c_u}{F_s} = \frac{500}{2} = 250\ \text{lb/ft}^2$$

From Figure 15.12, for $\beta = 56°$, $m = 0.185$. Thus, we have

$$H = \frac{c_d}{\gamma m} = \frac{250}{(100)(0.185)} = \mathbf{13.51\ ft}$$

# EXAMPLE 15.5

### Example 15.5

A cut slope is to be made in a soft saturated clay with its sides rising at an angle of 60° to the horizontal (Figure 15.17).
Given: $c_u$ – 40 kN/m² and $\gamma$ – 17.5 kN/m³.

a. Determine the maximum depth up to which the excavation can be carried out.
b. Find the radius, $r$, of the critical circle when the factor of safety is equal to 1 (Part a).
c. Find the distance $\overline{BC}$.

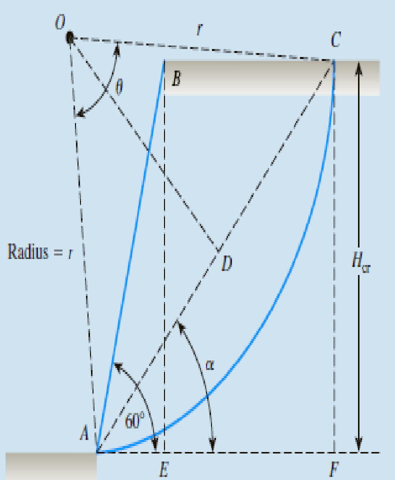



Figure 15.17

### Solution

**Part a**

Since the slope angle $\beta$ – 60° > 53°, the critical circle is a toe circle. From Figure 15.12, for $\beta$ = 60°, the stability number = 0.195.

$$H_{cr} = \frac{c_u}{\gamma m} = \frac{40}{17.5 \times 0.195} = \mathbf{11.72\ m}$$

**Part b**

From Figure 15.17,

$$r = \frac{\overline{DC}}{\sin\frac{\theta}{2}}$$

But

$$\overline{DC} = \frac{\overline{AC}}{2} = \frac{\left(\frac{H_{cr}}{\sin\alpha}\right)}{2}$$

so,

$$r = \frac{H_{cr}}{2\ \sin\alpha\ \sin\frac{\theta}{2}}$$

From Figure 15.13, for $\beta$ = 60°, $\alpha$ = 35° and $\theta$ = 72.5°. Substituting these values into the equation for $r$, we get

$$r = \frac{H_{cr}}{2\ \sin\alpha\ \sin\frac{\theta}{2}}$$

$$= \frac{11.72}{2(\sin 35)(\sin 36.25)} = \mathbf{17.28\ m}$$

**Part c**

$$\overline{BC} = \overline{EF} = \overline{AF} - \overline{AE}$$

$$= H_{cr}(\cot\alpha - \cot 60°)$$

$$= 11.72(\cot 35 - \cot 60) = \mathbf{9.97\ m}$$

# EXAMPLE 15.6

## Example 15.6

A cut slope was excavated in a saturated clay. The slope made an angle of 40° with the horizontal. Slope failure occurred when the cut reached a depth of 7 m. Previous soil explorations showed that a rock layer was located at a depth of 10.5 m below the ground surface. Assuming an undrained condition and $\gamma_{sat} = 18\ kN/m^3$, find the following.

a. Determine the undrained cohesion of the clay (use Figure 15.12).
b. What was the nature of the critical circle?
c. With reference to the toe of the slope, at what distance did the surface of sliding intersect the bottom of the excavation?

# EXAMPLE 15.6

## Solution

**Part a**

Referring to Figure 15.12,

$$D = \frac{10.5}{7} = 1.5$$

$$\gamma_{sat} = 18 \text{ kN/m}^3$$

$$H_{cr} = \frac{c_u}{\gamma m}$$

From Figure 15.12, for $\beta = 40°$ and $D = 1.5$, $m = 0.175$. So,

$$c_u = (H_{cr})(\gamma)(m) = (7)(18)(0.175) = \mathbf{22.05\ kN/m^2}$$

**Part b**

**Midpoint circle.**

**Part c**

Again, from Figure 15.14, for $D = 1.5$, $\beta = 40°$; $n = 0.9$. So,

$$\text{Distance} = (n)(H_{cr}) = (0.9)(7) = \mathbf{6.3\ m}$$