Applications of First Order Differential Equations

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Table of contents

Orthogonal Trajectories

Suppose that we have a family of curves given by

$$F(x, y, c) = 0. \tag{1}$$

One curve corresponding to each c in some range of values of the parameter c. In certain applications it is found desirable to know what curves have the property of intersecting a curve of the family (1) at right angles whenever they do intersect. That is, we wish to determine a family of curves with equation

$$G(x, y, k) = 0, \tag{2}$$

such that at any intersection of a curve of the family (2) with a curve of the family (1), the tangents to curves are perpendicular. The families (1) and (2) are said to be *orthogonal trajectories* of each other.

If two curves are to be orthogonal, then at each point of intersection the slopes of the curves must be negative reciprocals. Suppose that

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

is the differential equation corresponding to the family of curves (1), then the family of curves (2) is orthogonal to the family of curves (1) if it is a solution of the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)}. \tag{4}$$

That is equation (4) is the differential equation corresponding to the family of orthogonal trajectories (2).

Example

A family of straight lines with slope 2 is given by

$$y = 2x + c_1. \tag{5}$$

An orthogonal family of straight lines for *Eq* (1), will be

$$y = -\frac{1}{2}x + c_2. \tag{6}$$

Families of curve given in *Eq* (5) and *Eq* (6) are orthogonal to each other because

$$m_1 = 2 \quad \text{and} \quad m_2 = -\frac{1}{2},$$

$$m_1.m_2 = 2\left(-\frac{1}{2}\right) = -1.$$

Example

Show that the family of curves described by the family of concentric circles

$$x^2 + y^2 = c_1, \tag{7}$$

and the family of straight lines passing through the origin

$$y = mx, \tag{8}$$

are orthogonal.

Solution.
The differential equation corresponding to the family of curves (7) is

$$\frac{dy}{dx} = -\frac{x}{y}, \quad x \neq 0, \quad y \neq 0. \tag{9}$$

The differential equation corresponding to the family of curves (8) is given by

$$\frac{dy}{dx} = \frac{y}{x}, \quad x \neq 0. \tag{10}$$

By solving *Eq* (10) we get

$$y = c_2 x, \ c_2 \neq 0. \tag{11}$$

The family of curves (11) is a family of straight lines passing through the origin which is orthogonal to the family of curves (7).

Note If two lines are orthogonal, then the product of their slopes is equal to -1.

$$m_1.m_2 = -1.$$

If slope of the tangent line to one curve is

$$m_1 = \frac{dy}{dx} = f(x, y),$$

then the slope the orthogonal trajectory will be

$$m_2 = \frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

Example

Find the orthogonal trajectories of the circles

$$x^2 + (y - c)^2 = c^2, \; c \neq 0. \tag{12}$$

Solution.

The differential equation of the given family can be obtained by differentiating it with respect to x,

$$2x + 2(y - c)\frac{dy}{dx} = 0. \tag{13}$$

From *Eq* (12) we find value of c, and substitute in *Eq* (13)

$$x^2 + y^2 - 2yc + c^2 = c^2,$$
$$2yc = x^2 + y^2,$$

$$c = \frac{x^2 + y^2}{2y},\ y \neq 0,$$

$$2x + 2(y - \frac{x^2 + y^2}{2y})\frac{dy}{dx} = 0,$$

$$x + (\frac{-x^2 + y^2}{2y})\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - x^2} = f(x, y), \quad y \neq \pm x. \tag{14}$$

The differential equations of the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \Longrightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}, \; x \neq 0. \tag{15}$$

The solution of the differential equation (15) can be obtained by reducing the differential equation to separable variable form with $v = \frac{y}{x}$, is

$$(x - c)^2 + y^2 = c^2. \quad (16)$$

Eq (16) is the equation of the orthogonal trajectories.

Example

Find the orthogonal trajectories to the family

$$c_1 x^2 - y^2 = 1, \; c_1 \neq 0. \tag{17}$$

Solution.

The differential equation of the given family can be obtained by differentiating *Eq* (17) with respect to x,

$$2c_1 x - 2y\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = c_1 \frac{x}{y}, \; y \neq 0. \tag{18}$$

From *Eq* (17), we find value of c_1 and substitute in *Eq* (18)

$$c_1 = \frac{1+y^2}{x^2}, \; x \neq 0,$$

$$\frac{dy}{dx} = \frac{1+y^2}{xy}. \tag{19}$$

The differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\frac{xy}{1+y^2}.$$

Solving the differential equation by separating variables

$$\frac{1+y^2}{y}dy = -xdx,$$

integrating both sides

$$\int \frac{1+y^2}{y}dy = -\int xdx,$$

$$\int \left(\frac{1}{y}+y\right)dy = -\int xdx,$$

$$\ln|y| + \frac{y^2}{2} = -\frac{x^2}{2} + c,$$

$$\ln y^2 + y^2 = -x^2 + c_2. \tag{20}$$

Eq (20) is the family of curves which is orthogonal to family of curves (17).

Example

Find the orthogonal trajectories to the family

$$y^2 = cx^3. \tag{21}$$

Solution.

The differential equation of the given family can be obtained by differentiating *Eq* (21) with respect to x.

$$2yy' = 3cx^2. \tag{22}$$

From *Eq* (21) we find value of c and substitute in *Eq* (22)

$$2yy' = 3\frac{y^2}{x^3}x^2 = 3\frac{y^2}{x}, \quad x \neq 0 \quad \& \; y \neq 0,$$

we obtain

$$\frac{dy}{dx} = \frac{3y}{2x} \; = \; f(x, y). \tag{23}$$

The Differential equation of the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} = -\frac{2x}{3y}.$$

Solving of the differential equation by separating variables

$$3ydy = -2xdx,$$

integrating both the sides

$$\int 3ydy = -2\int xdx,$$

$$\frac{3y^2}{2} = -x^2 + c,$$

then the family curves of orthogonal trajectories to the family (21) is

$$3y^2 + 2x^2 = c_1, \text{ where } c_1 = 2c \neq 0.$$

Example

Find the orthogonal trajectories to the family

$$x^3 + 3xy^2 = c \ ; \ c \neq 0. \tag{24}$$

Solution.

The differential equation of the given family can be obtained by differentiating *Eq* (24) with respect to x.

$$3x^2 + 3y^2 + 3x.2y\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy} = f(x, y),\ x \neq 0 \ \text{ and } y \neq 0.$$

The differential equation of the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} = \frac{2xy}{x^2 + y^2}. \tag{25}$$

Eq (25) is homogeneous differential equation

$$\left(x^2 + y^2\right) dy - 2xydx = 0.$$

Substituting

$$x = vy, dx = vdy + ydv,$$

in the differential equation

$$\begin{aligned}
\left(v^2y^2+y^2\right)dy-2vyy\left(vdy+ydv\right) &= 0\\
v^2y^2dy+y^2dy-2v^2y^2dy-2vy^3dv &= 0\\
y^2dy-v^2y^2dy-2vy^3dv &= 0\\
y^2\left(1-v^2\right)dy &= 2vy^3dv\\
\frac{dy}{y} &= \frac{2vdv}{1-v^2};\ y\neq\pm x,
\end{aligned}$$

integrating both the sides

$$\int\frac{dy}{y}=\int\frac{2vdv}{1-v^2},$$

$$\begin{aligned}\ln|y| &= -\ln\left|1 - v^2\right| + c_1 \\ \ln|y| &= -\ln\left|1 - \frac{x}{y^2}^2\right| + \ln c_2 \quad \text{using } c_1 = \ln c_2\end{aligned}$$

$$(y^2 - x^2) = c_3 y, \;\text{ where } c_3 = \pm c_2. \tag{26}$$

Then *Eq* (26) represents the family of orthogonal trajectories.

Example

Find the member of the orthogonal trajectories for

$$x^2 = cy \quad ; c \neq 0. \tag{27}$$

that passes through $(1, 1)$.

Solution.

The differential equation of the given family can be obtained by differentiating *Eq* (27) with respect to x.

$$2x = c\frac{dy}{dx}. \tag{28}$$

From *Eq* (27) we find value of c, and substitute in *Eq* (28)

$$c = \frac{x^2}{y},$$

$$\frac{dy}{dx} = 2x.\frac{y}{x^2} = \frac{2y}{x} = f(x,y),\ x \neq 0 \text{ and } y \neq 0.$$

The differential equation of the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\frac{x}{2y}, \tag{29}$$

Eq (29) is a differential equation of separable form

$$2ydy = -xdx$$

$$\begin{aligned} 2\int ydy &= -\int xdx \\ y^2 &= -\frac{x^2}{2} + c \end{aligned}$$

$$2y^2 + x^2 = c_1,\ c_1 = 2c. \tag{30}$$

Eq (30) is the family of orthogonal trajectories. To find the specific orthogonal trajectory, put $x = 1$ and $y = 1$ in *Eq* (30), we find $c_1 = 3$, then $2y^2 + x^2 = 3$ is the member of orthogonal trajectories passing through $(1, 1)$.

Example

Compute the orthogonal trajectories of the family of curves

$$x^2 + 2xy - y^2 + 4x - 4y = c. \tag{31}$$

Solution.

We should first find the differential equation from the *Eq* (31) by differentiating with respect to x, we obtain

$$2x + 2y + 2xy' - 2yy' + 4 - 4y' = 0,$$

and so

$$y' = -\frac{x+y+2}{x-y-2} = f(x,y),\ x-y-2 \neq 0.$$

Thus, the slope of the orthogonal trajectories of *Eq* (31) is

$$y' = \frac{x-y-2}{x+y+2},\ x+y+2 \neq 0. \tag{32}$$

Eq (32) can be written in the form

$$(-x+y+2)dx + (x+y+2)dy = 0,$$

this equation is an exact differential equation can be written in the form

$$(-xdx + ydy) + 2dx + 2dy + (ydx + xdy) = 0,$$

or

$$d\left[\frac{1}{2}(y^2 - x^2) + 2(x+y) + xy\right] = 0.$$

Thus, the general solution of *Eq* (32) is

$$x^2 - 2xy - y^2 - 4x - 4y = c_1. \qquad (33)$$

The curves given by *Eq* (33) are the orthogonal trajectories of *Eq* (31) .Both of these families consist of hyperbolas.

Exercises

Exercise 1 :
Find a first order differential equation for the given family of curves.

1. $y(y^2 + x^2) = c$.
2. $\ln|xy| = c\left(x^2 + y^2\right)$.
3. $y = e^{x^2} + ce^{-x^2}$.
4. $y = \sin x + ce^x$.
   Find the orthogonal Trajectories of the family of curves.
5. $y^2 = x - c$.

1. $y^2 = c_1 x^3$.
2. $2x^2 + y^2 = c^2$.
3. $y = x + ce^{-x}$.
4. $x^2 + 2y^2 = c^2$.
5. $xye^{x^2} = c$.
6. $c_1 x^2 - y^2 = 1$.
7. $y^2 = c\left(1 - x^2\right)$.
8. $2x^2 + y^2 = 6cx$.

1. $2y + x + c_1 e^{2y} = 0.$
2. $y = x + 2 - c_1 e^{y}.$
3. $x^2 - y^2 = cx.$
4. $y^2 = \frac{x^3}{(a-x)}$ (The cissoid).
5. $y(x^2 + c) + 2 = 0.$
6. $y^2 = ax^2(1 - cx)$, with a held fixed.

1. Find the member of the orthogonal trajectories for $x^2 = c$ that passes through $(1, 1)$.
2. Find the member of the orthogonal trajectories for $x^2 + 3y^2 = cy$ that passes through $(1, 2)$.

Growth and Decay

Many natural processes involve quantities that increase or decrease at a rate proportional to the amount of the quantity present.
If $y = y(t)$ denotes the value of a quantity y at any time t, and if y changes at the rate proportional to amount present.

$$\frac{dy}{dt} = ky, \tag{34}$$

where k is a constant of proportionality. *Eq* (34) describes the growth if $k > 0$ or decay if $k < 0$.

Note If at some initial time quantity is known and is $y(t_0) = y_0$, then the differential equation

$$\frac{dy}{dt} = ky,$$

and the initial value $y(t_0) = y_0$ make it an initial value problem. To solve the initial value problem

$$\begin{cases} \frac{dy}{dt} = ky, \\ y(t_0) = y_0. \end{cases} \tag{35}$$

We have

$$\begin{aligned} \frac{dy}{dt} &= ky \\ \int \frac{dy}{y} &= \int kdt \Rightarrow \ln|y| = kt + \ln c \\ y(t) &= c_1 e^{kt},\ c_1 = \pm c. \end{aligned}$$

Using the initial condition $t_0 = 0, \quad y = y_0,\ c_1 e^0 = y_0,\ y_0 = c_1$
$y(t) = y_0 e^{kt}$.
Additional constant k can be found by using additional condition given in the problem.

Example

A certain culture of bacteria grows at a rate proportional to its size. If the size doubles in 4 days, find the time required for the culture to increase to 10 times its original size.

Solution.

Let $P(t)$ be the size of the culture after t days, then

$$\begin{cases} \frac{dp}{dt} = kp, \\ p(0) = p_0. \end{cases}$$

The initial condition is used is to find arbitrary constant c, the additional condition is used to find the constant k

$$p(4) = 2p_0$$

$$\begin{aligned} \frac{dp}{dt} &= kp \\ \int \frac{dp}{p} &= \int k dt \\ \ln p &= kt + \ln c \\ p(t) &= c_1 e^{kt},\ c_1 = \pm c. \end{aligned}$$

By using the initial condition at $t = 0$, $p(0) = p_0$, $c_1 = p_0 p(t) = p_0 e^{kt}$. Using an additional condition to find k, $t = 4$, $p(4) = 2p_0$

$$\begin{aligned} 2p_0 &= p_0 e^{4k} \\ e^{4k} &= 2 \Rightarrow k = \frac{\ln 2}{4} \\ p &= p_0 e^{\frac{\ln 2}{4} t}. \end{aligned}$$

Time required for the culture to increase to 10 times its original size

$$
\begin{aligned}
10p_0 &= p_0 e^{\frac{\ln 2}{4}t} \\
e^{\frac{\ln 2}{4}t} &= 10 \\
\frac{\ln 2}{4}t &= \ln 10 \Rightarrow t = \frac{4\ln 10}{\ln 2} \\
t &\simeq 13.29 \text{ days.}
\end{aligned}
$$

Example

The population of a town grows at a rate proportional to the population at any time. Its initial population of 1000 increases by 10% in 5 years. What will be the population after 50 years?

Solution.

Let $P = P(t)$ be the population at time t

$$\frac{dP}{dt} = kP.$$

Initial condition

$$P(0) = 1000.$$

Additional condition, increase of initial population by 10% in 5 years,

$$P(5) = \frac{110}{100} X 1000 = 1100$$

$$\begin{aligned} \frac{dP}{P} &= kdt \\ \int \frac{dP}{P} &= \int kdt \\ \ln P &= kt + \ln c \\ P(t) &= c_1 e^{kt},\ c_1 = \pm c. \end{aligned}$$

Invoking the initial condition $P(0) = 1000$, to find

$$\begin{aligned} c_1 &= 1000 \\ P(t) &= 1000 e^{kt}. \end{aligned}$$

Using additional condition to find k, at $t = 5$, $p(5) = 1100$,

$$\begin{aligned}
1100 &= 1000e^{5k} \\
e^{5k} &= \frac{1100}{1000} = 1.1 \\
5k &= \ln 1.1 \Rightarrow k = \frac{\ln 1.1}{5} \\
P(t) &= 1000e^{\frac{\ln 1.1}{5}t}
\end{aligned}$$

After 50 years the population will be

$$\begin{aligned}
P(50) &= 1000e^{\frac{\ln 1.1}{5}X50} \\
P(50) &\simeq 2586.
\end{aligned}$$

Note 1. The rate of growth is proportional to the amount present at any time can be described by differential equation

$$\frac{dP}{dt} = kP$$

where k is constant of proportionality P is the amount present at any time t.

Note 2. The rate of growth is proportional to square of the amount present at any time can be described by the differential equation.

$$\frac{dP}{dt} = kP^2,$$

where k is constant of proportionality P is the amount present at any time t.

Radio Active Decay

The rate of decay of radio active material is proportional to the amount present at any time can be described by differential equation.

$$\frac{dy}{dt} = ky,$$

where k is constant of proportionality y is the amount present at any time t.

Half-life. It is the time necessary for half of the radio active material to decay.

Example

A radio active material has an initial mass 100 mg. After two years it is left to 75 mg. Find the amount of the material at any time. What is the period of its half-life?

Solution. Let $y(t)$ be the amount of material present at any time t

$$\frac{dy}{dt} = ky,$$

the initial condition $y(0) = 100$ mg, and the additional condition at $t = 2$, $y(2) = 75$ mg, yield

$$\begin{aligned}\frac{dy}{dt} &= ky\\ \frac{dy}{y} &= kdt\\ \int \frac{dy}{y} &= \int kdt\\ \ln|y| &= kt + \ln c\\ y(t) &= c_1 e^{kt},\ c_1 = \pm c.\end{aligned}$$

Using initial condition, we obtain $c_1 = 100$

$$y(t) = 100e^{kt}.$$

Additional condition is used to obtain k

$$\begin{aligned} 75 &= 100e^{2k} \\ e^{2k} &= \frac{75}{100} = 0.75 \\ 2k &= \ln 0.75 \\ k &= \frac{1}{2}\ln 0.75 \\ y(t) &= 100e^{\frac{1}{2}\ln 0.75t}. \end{aligned}$$

Half-life of the material is time for $y = 50mg$

$$\begin{aligned} 100e^{\frac{1}{2}\ln 0.75t} &= 50 \\ e^{\frac{1}{2}\ln 0.75t} &= \frac{50}{100} \\ \frac{1}{2}\ln 0.75t &= \ln 0.5 \\ t &= \frac{2\ln(0.5)}{\ln 0.75} \simeq 4.82.\ \text{years} \end{aligned}$$

Example

In 1980 the department of natural resources released 1000 splake (a crossbreed of fish) into lake. In 1987 the population of splake in the lake was estimated to be 3000. Estimate the population of splake in the lake in the year 2010.

Solution.

We have

$$\frac{dP}{dt} = kP,$$

then

$$P(t) = c_1 e^{kt}.$$

Since $P(0) = 1000$, and $P(7) = 3000$, then

$$P(t) = 1000e^{kt},$$

$$P(7) = 3000 = 1000e^{7k}$$

$$k = \frac{\ln 3}{7}$$

$$P(t) = 1000e^{\frac{\ln 3}{7}t}.$$

After 30 years we have

$$P(30) = 1000e^{\frac{\ln 3}{7}(30)} \simeq 111052.$$

Example

Initially there were 100 milligrams (mg.) of a radioactive substance present after 6 hours the mass decreased by %3 .If the rate of decay is proportional to the amount of substance present at any time. Find the amount remaining after 24 hours. Determine the half-life of radioactive substance.

Solution.

We denote by $A(t)$ to the substance present at time t.From the differential equation $\frac{dA(t)}{dt} = kdt$. We deduce $A(t) = ce^{kt}$. But $A(0) = 100$ and $A(6) = 100 - 3 = 97$, then $A(t) = 100e^{kt}$.

$$A(6) = 100e^{6k} \implies k = \frac{\ln(0.97)}{6}.$$

Thus

$$A(t) = 100e^{-\frac{\ln(0.97)}{6}t},$$

hence

$$A(24) = 100e^{-\frac{\ln(0.97)}{6}(24)} \simeq 88.48 \ mg.$$

For the half-life, we have $A(t) = \frac{100}{2} = 50 \ mg$. and

$$50 = 100e^{-\frac{\ln(0.97)}{6}t} \implies t = \frac{-6\ln 2}{\ln(0.97)} \simeq 136 \text{ hours.}$$

So the half -life of radioactive substance is 136 hours.

Example

Initial population P_0 of a country doubles after 50 years. In how many years, the population will be $3P_0$?, if the rate of growth of population is directly proportional to population at any time t.

Solution.
We denote $P(t)$ to the population at time t. Then we have the differential equation

$$\frac{dP}{dt} = kP(t),$$

which has the solution

$$P(t) = ce^{kt}.$$

But $P(0) = P_0$ and $P(50) = 2P_0$, hence

$$P(t) = P_0 e^{kt}$$

$$2P_0 = P_0 e^{50k} \implies k = \frac{\ln 2}{50}.$$

So we can compute the population from the following formula

$$P(t) = P_0 e^{\frac{\ln 2}{50} t}$$

$$3P_0 = P_0 e^{\frac{\ln 2}{50} t} \implies t = \frac{50 \ln 3}{\ln 2} \simeq 78.5 \text{ years}.$$

Thus, after 78.5 year the population will be $3P_0$.

Example

A radioactive substance has a half life of 1620 years. How many grams of a sample of 120 grams of this substance will be left after 100 years ?.

Solution.

We denote by $m(t)$ the mass of radioactive substance at time t. Then we have the differential equation

$$\frac{dm}{dt} = km(t),$$

which has the solution

$$m(t) = ce^{kt}.$$

But $m(0) = 120$, and $m(1620) = \frac{120}{2} = 60$. Then

$$m(t) = 120e^{kt},$$

implies

$$m(1620) = 60 = 120e^{1620k} \implies k = \frac{-\ln 2}{1620},$$

hence the mass of radioactive substance at time t satisfies the following equation

$$m(t) = 120e^{\frac{-\ln 2}{1620}t}.$$

Thus,

$$m(100) = 120e^{\frac{-\ln 2}{1620}(100)} \simeq 115 \text{ grams}.$$

So, after 100 years the mass m will be 115 grams.

Example

The population of a town is presently 38300. The town grows at a rate of 1.2% per year. Find the number of years it takes the population to grow to 42500.

Solution.

The population of the town satisfies the differential equation

$$\frac{dP}{dt} = kP(t),$$

then

$$P(t) = ce^{kt}.$$

But $P(0) = 38300$ and $k = 0.012$. Then

$$P(t) = 38300e^{0.012t},$$

if $P = 42500$,we have

$$42500 = 38300e^{0.012t} \implies t = \frac{1}{0.012}\ln(\frac{425}{383}) \simeq 9 \text{ years.}$$

Thus, after 9 years the population of the town will be 42500.

Example

The sum of 5000 $ is invested at a rate of 8% per year compounded contiguously. What will the amount be after 25 years?.

Solution.
Let $y(t)$ be the amount of money (capital plus interest) at time t. The the rate of change of the money at time t is given by

$$\frac{dy}{dt} = \frac{8}{100}y,$$

this equation has the solution

$$y(t) = ce^{(\frac{8}{100})t}.$$

But $y(0) = 5000$ (the initial amount invested), we find

$$y(t) = 5000e^{(\frac{8}{100})t},$$

we find

$$y(25) = 5000e^{(\frac{8}{100})25} = 5000e^{2} \simeq \$36,945.28.$$

Thus, the amount of money will be $36,945.28$ \$ after 25 years.

Newton's Law of Cooling

The rate of change of temperature $T = T(t)$ of a body at time t is proportional to the difference $T - T_s$, temperature of the body and T_s temperature of its surrounding. It can be described it in term of first order differential equation

$$\frac{dT}{dt} = k(T - T_s), \tag{36}$$

where k is a constant of proportionality.

The temperature of body can be identified, as it changes with time, whereas the temperature of surrounding is not changing with time. We have a differential equation of first order, then

$$\begin{aligned}
\frac{dT}{dt} &= k(T - T_s) \\
\frac{dT}{T - T_s} &= kdt \\
\int \frac{dT}{T - T_s} &= \int kdt \\
\ln |T - T_s| &= kt + \ln c \\
T - T_s &= c_1 e^{kt},\ c_1 = \pm c \neq 0
\end{aligned}$$

$$T(t) = T_s + c_1 e^{kt}. \tag{37}$$

The constant of integration c_1 can be calculated by using the initial condition and the constant of proportionality k can be calculated by using an additional condition.
Note. In *Eq* (37), if $c_1 > 0$ then $T(t) > T_s$, in this case we have Newton's Law of cooling. But if $c_1 < 0$ then $T(t) < T_s$ and we have Newton's law of heating.

Example

A glass of hot water whose initial temperature is $80^0\ C$ is placed in a room where the temperature is $30^0 C$. After one minute the water temperature drops to $70^0 C$. What will be the temperature after 3 minutes? At what time the water cools down to $40^0\ C$.

Solution.

Newton's law of cooling is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

as $T_s = 30^0 C$, $T(0) = 80^0 C$, and $T(1) = 70^0 C$, then

$$\frac{dT}{dt} = k(T - 30),$$

and the general solution of the differential equation is

$$T(t) = 30 + c_1 e^{kt}.$$

From the initial condition and additional condition we have

$$\begin{aligned} 30 + c_1 &= 80 \\ c_1 &= 80 - 30 = 50 \\ T(t) &= 30 + 50e^{kt} \end{aligned}$$

$$\begin{aligned} 30 + 50e^k &= 70 \\ e^k &= \frac{40}{50} \\ k &= \ln(0.8), \end{aligned}$$

$$T(t) = 30 + 50e^{\ln(0.8)t}. \tag{38}$$

Eq (38) gives the temperature of the water at any time t. When $t = 3$ the temperature of water will be

$$\begin{aligned} T(3) &= 30 + 50e^{\ln(0.8)3}, \\ T(3) &= 30 + 25.6 = 55.6^0 C. \end{aligned}$$

Now we have to find the time for the water to cool down to $40^0 C$ will be

$$\begin{aligned} 30 + 50e^{\ln(0.8)t} &= 40 \\ 50e^{\ln(0.8)t} &= 10 \\ e^{\ln(0.8)t} &= \frac{10}{50} \\ \ln(0.8)t &= \ln(0.2) \\ t &= \frac{\ln(0.2)}{\ln(0.8)} \\ t &\simeq 7.2\,\text{min}. \end{aligned}$$

Example

A thermometer, which has been at the reading 70^0F inside a room, is placed outside air temperature 10^0F. After 3 minutes it is found that the thermometer is at 25^0F. What will be the thermometer reading after 7minutes?

Solution.

Newton's law of cooling is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s).$$

But $T_s = 10^0F$, $T(0) = 70^0F$ (initial condition), and $T(3) = 25^0F$ (additional condition) then

$$\frac{dT}{dt} = k(T - 10),$$

and the general solution of the differential equation is

$$T(t) = 10 + ce^{kt}.$$

Using the initial condition, to obtain c

$$\begin{aligned} 10 + c &= 70 \\ c &= 70 - 10 = 60 \\ T(t) &= 10 + 60e^{kt}. \end{aligned}$$

Using the additional condition, to obtain k

$$\begin{aligned} 10 + 60e^{3k} &= 25 \\ e^{3k} &= \frac{15}{60} = 0,25 \\ k &= \frac{\ln(0.25)}{3}, \end{aligned}$$

hence

$$T(t) = 10 + 60e^{\frac{\ln(0.25)}{3}t}. \tag{39}$$

Eq (39) gives the temperature of the thermometer at time t.
When $t = 7$ the temperature of the thermometer will be

$$\begin{aligned} T(7) &= 10 + 60e^{\frac{\ln(0.25)}{3}7} \\ T(7) &\simeq 12.36^0 F. \end{aligned}$$

Example

A pizza is removed from an oven, its temperature is measured to be $350^0 F$. Four minutes later its temperature is $250^0 F$. How long it will take for the pizza to cool off to $150^0 F$, if the room temperature is $80^0 F$?

Solution.

Newton's law of cooling is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s).$$

But $T_s = 80^0 F$, $T(0) = 350^0 F$ (Initial condition) and $T(4) = 250^0 F$ (additional condition), then

$$\frac{dT}{dt} = k(T - 80),$$

and the general solution of the differential equation is

$$T(t) = 80 + ce^{kt}.$$

Using the given initial condition, to obtain c

$$\begin{aligned} 80 + c &= 350 \\ c &= 350 - 80 = 270 \\ T(t) &= 80 + 270e^{kt}. \end{aligned}$$

Using additional condition, to obtain k

$$\begin{aligned} 80 + 270e^{4k} &= 250 \\ k &= \frac{1}{4}\ln(\frac{17}{27)}) \end{aligned}$$

$$T(t) = 80 + 270e^{\frac{1}{4}\ln(\frac{17}{27)})t}. \quad (40)$$

Eq (40) gives the temperature of the pizza at time t. Now the time for pizza to cool off to $150^0 F$ is

$$\begin{aligned} 80 + 270e^{\frac{1}{4}\ln(\frac{17}{27})t} &= 150 \\ 270e^{\frac{1}{4}\ln(\frac{17}{27})t} &= 70 \\ e^{\frac{1}{4}\ln(\frac{17}{27})t} &= \frac{7}{27} \\ t &= \frac{4\ln(\frac{7}{27})}{\ln(\frac{17}{27})} \simeq 11.66\,\text{min}, \text{ or } 12.06. \end{aligned}$$

Thus, the time for the pizza to cool off to $150^0 F$ is 12 min. and 06 sec.

Example

A copper bar was heated to $150^0 C$, then left in air having temperature $25^0 C$. After 5 minutes the temperature of the bar was reduced to half its initial value. When the temperature of the bar reach $30^0 C$?

Solution.

Newton's law of cooling is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

and the solution of the differential equation is

$$T(t) = T_s + c_1 e^{kt}.$$

But $T(0) = 150$, $T_s = 25$ and $T(5) = 75$, then we have

$$T(t) = 25 + 125e^{kt},$$

$$T(5) = 75 = 25 + 125e^{5k} \implies k = \frac{1}{5}\ln(\frac{2}{5}).$$

Then,the temperature of the bar at time t is given by

$$T(t) = 25 + 125e^{\frac{1}{5}\ln(\frac{2}{5})t}.$$

Now if $T(t) = 30$, then

$$30 = 25 + 125e^{\frac{1}{5}\ln(\frac{2}{5})t} \implies t \simeq \frac{-5\ln 25}{\ln(\frac{2}{5})} \simeq 17.6 \text{ minutes}.$$

Thus, after 17.6 minutes the temperature of the bar reaches to 30^0C.

Example

A thermometer is taken from inside room to outside where air temperature is $4^0 F$. After 1 minute the temperature reads $64^0\ F$. After 8 minutes, it reads $20^0 F$. What was the initial temperature of the room?

Solution.

Newton's law of cooling is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

and the solution of the differential equation is

$$T(t) = T_s + c_1 e^{kt}.$$

But, $T_s = 4$, $T(1) = 64$ and $T(8) = 20$, then

$$T(t) = 4 + c_1 e^{kt},$$

$$T(8) = 20 = 4 + c_1 e^{8k} \implies 16 = c_1 e^{8k}. \tag{42}$$

From (41) and (42) we have

$$\frac{c_1 e^{8k}}{c_1 e^{k}} = \frac{16}{60} \implies e^{7k} = \frac{4}{15}$$

$$k = \frac{\ln(\frac{4}{15})}{7} \implies k \simeq -0.189,$$

$$c_1 = 60e^{-k} \simeq 60e^{0.189} \simeq 72.48.$$

Thus, the temperature of the thermometer is given by

$$T(t) = 4 + 72.48e^{-0.189t}. \tag{43}$$

So the initial temperature of the room is
$T(0) = 4 + 72.48 = 76.48^0 F$.

Example

A thermometer reading 70^0F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer reads that the thermometer reads 110^0F after $\frac{1}{2}$ minute, and 145^0F after one minute. How hot is the oven?

Solution.

Newton's law of heating is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

and the solution of the differential equation is

$$T(t) = T_s + c_1 e^{kt}.$$

But $T(0) = 70$, then

$$70 = T_s + c_1 \quad \Longrightarrow \quad c_1 = 70 - T_s.$$

$$T(t) = T_s + (70 - T_s)e^{kt}.$$

Here $T_s > 70$. At $t = \frac{1}{2}$, we have $T(\frac{1}{2}) = 110$, then

$$110 = T_s + (70 - T_s)e^{\frac{k}{2}}, \tag{44}$$

and at $t = 1$ we have $T(1) = 145$, hence

$$145 = T_s + (70 - T_s)e^{k}. \tag{45}$$

From *Eq* (44) and *Eq* (45) we deduce that

$$\frac{110 - T_s}{70 - T_s} = e^{\frac{k}{2}} \quad \frac{145 - T_s}{70 - T_s} = e^k,$$

$$\frac{(110 - T_s)^2}{(70 - T_s)^2} = \frac{145 - T_s}{70 - T_s},$$

$$12100 + T_s^2 - 220T_s = 10150 - 215T_s + T_s^2,$$

$$1950 = 5T_s \implies T_s = 390^0 F.$$

We can find the value of k from

$$T(1) = 145 = 390 - 320e^{k} \implies k = \ln(\frac{245}{320}).$$

Thus, the temperature inside of the oven is given the following formula at time t

$$T(t) = 390 - 320e^{\ln(\frac{245}{320})t}.$$

Example

A steel body is taken from inside room which has the temperature $20\ ^0C$, and placed inside an oven its temperature $100\ ^0C$. After one minute the temperature of the body becomes 60^0C. Find the temperature of the body after 3 minutes.

Solution.

Newton's law of heating is described by the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

and the solution of the differential equation is

$$T(t) = T_s + c_1 e^{kt}.$$

But $T(0) = 20,\ T(1) = 60$ and $T_s = 100$, then

$$T(t) = 100 + e^{kt},$$

$$T(0) = 20 = 100 + c_1 \implies c_1 = -80,$$

$$T(1) = 60 = 100 - 80e^k \implies k = -\ln 2.$$

Thus, the temperature of the body is given by the formula at time t

$$T(t) = 100 - 80e^{-(\ln 2)t}.$$

The temperature of the body after 3 minutes is

$$T(3) = 100 - 80e^{-3\ln 2} = 100 - 80e^{\ln(\frac{1}{8})} = 100 - \frac{80}{8} = 90^0 C.$$

Example

When a cake is removed from an oven, its temperature is measured at $300^0 F$. Three minutes later its temperature is $200^0 F$. How long the cake to cool off a room of $70^0 F$.

Solution.

The temperature of the cake is given by the formula at time t

$$T(t) = T_s + c_1 e^{kt}. \tag{46}$$

But $T(0) = 300$, $T_s = 70$, and $T(3) = 200$ then

$$T(t) = 70 + 230e^{kt},$$

$$T(3) = 200 = 70 + 230e^{3k} \implies k = \frac{1}{3}\ln(\frac{13}{23}).$$

Thus

$$T(t) = 70 + 230e^{\frac{1}{3}\ln(\frac{13}{23})t}. \tag{47}$$

We note that (12) furnishes no finite solution to $T(t) = 70$, since $\lim_{t\to\infty} T(t) = 70$. Yet we intuitively expect the cake to reach room temperature after reasonably long period of time t.

Example

According to Newton's law of cooling, if the temperature of the air is $300^0 C$ and the substance cools from $370^0 C$ to $340^0 C$ in 15 minutes, find when the temperature will be $310^0 C$.

Solution.

The differential equation is given by

$$\frac{dT}{dt} = k(T - T_s),$$

then the temperature of the substance is given by the formula

$$T(t) = T_s + c_1 e^{kt}.$$

But $T_s = 300$, $T(0) = 370$, and $T(15) = 340$, then

$$T(0) = 300 + c_1 = 370 \implies c_1 = 70,$$

$$T(t) = 300 + 70e^{kt},$$

$$T(15) = 340 = 300 + 70e^{15t} \implies k = \frac{1}{15}\ln(\frac{4}{7}).$$

So

$$T(t) = 300 + 70e^{\frac{1}{15}\ln(\frac{4}{7})t}.$$

$$300 + 70e^{\frac{1}{15}\ln(\frac{4}{7})t} = 310 \implies t = \frac{-15\ln 7}{\ln(\frac{4}{7})} \simeq 52.6 \text{ minutes.}$$

Hence the temperature of the substance reaches to 310^0C after 52.6 minutes.