Linear Differential Equations of Higher Order

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Existence Theorem and Fundamental Set of Solutions

Definition

The general linear differential equation of order n is an equation that can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = R(x), \quad (1)$$

where R and the coefficients $a_1, a_2, \ldots, a_n$ are functions of x defined on an interval I. The equation (1) is called a homogeneous linear differential equation if the function $R(x)$ is zero for all $x \in I$. Suppose that the coefficients $a_1, a_2, \ldots, a_n$ and the function R are continuous on an interval I such that $a_n(x)$ is never zero on I, then the equation (1) is said to be normal on I. If R is not equal to zero on I, the equation (1) is called non - homogeneous linear differential equation.

Example

$$\frac{d^2y}{dt^2} + w^2y = 0 \quad \text{(undamped free vibration).}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E_0\cos(wt) \qquad (\ LRC - \text{circuit}).$$

$$x^2y'' + xy' + \lambda^2 y = 0 \qquad (\text{ Bessel differential equation }).$$

Now we suppose that y_1, y_2,. . . , y_k are solutions of the homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0, \qquad (2)$$

then for all for all $c_1, c_2, \ldots, c_k$ in $\mathbb{R}$

$$y = c_1y_1 + c_2y_2 + \ldots + c_ky_k,$$

is also a solution of (2).

So we have the following theorem

Theorem (Linear combination)

Any linear combination of solutions of a homogeneous linear differential equation is also a solution.

Now we give the existence and uniqueness theorem for an initial value problem (IVP) for nth-order linear differential equation.

Theorem (Existence Theorem)

Given an nth-order linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = R(x). \quad (3)$$

that is normal on an interval I. Suppose $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are n arbitrary real numbers. Then there exists a unique solution $y = y(x)$ of (3) satisfying the initial conditions

$$y(x_0) = y_0, \; y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}. \quad (4)$$

Example

Discuss the existence of unique solution of (IVP)

$$\begin{cases} (x^2+1)y'' + x^2y' + 5y = \cos(x) \\ y(3) = 2, \quad y'(3) = 1. \end{cases}$$

Solution.
The functions

$$a_2(x) = x^2 + 1, a_1(x) = x^2, a_0(x) = 5,$$

and

$$R(x) = \cos(x).$$

are continuous on $I = \mathbb{R} = (-\infty, +\infty)$, and $a_2(x) \neq 0$ for all $x \in \mathbb{R}$, the point $x_0 = 3 \in I$. Then Theorem (4) assures that the IVP has a unique solution on $\mathbb{R}$.

Example

Find an interval I for which the initial values problem (IVP)

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0, \\ y(1) = 0, \quad y'(1) = 1. \end{cases}.$$

has a unique solution around $x_0 = 1$.

Solution. The function $a_2(x) = x^2$, is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x > 0$ or $x < 0$. But $x_0 = 1 \in I_1 = (0, \infty)$. The function $a_1(x) = \frac{x}{\sqrt{2-x}}$, is continuous on $I_2 = (-\infty, 2)$ and the function

$a_0(x) = \dfrac{2}{\sqrt{x}}$, is continuous on $I_1 = (0, \infty)$. Then the (IVP) has a unique solution on $I_1 \cap I_2 = (0, 2) = I$. We can take any interval $I_3 \subset (0, 2)$ such that $x_0 = 1 \in I_3$. So I is that the largest interval for which the (IVP) has a unique solution.

Example

Find an interval I for which the *IVP*

$$\begin{cases} (x-1)(x-3)y'' + xy' + y = x^2, \\ y(2) = 1, \quad y'(2) = 0. \end{cases}$$

has a unique solution about $x_0 = 2$.

Solution.

The functions

$$a_2(x) = (x-1)(x-3) \;\; a_1(x) = x \, a_0(x) = 1 \, R(x) = x^2,$$

are continuous on $\mathbb{R}$. But $a_2(x) \neq 0$ if $x \in (-\infty, 1)$ or $x \in (1, 3)$ or $x \in (3, \infty)$. As $x_0 = 2$ so we take $I = (1, 3)$. Then the *IVP* has a unique solution on $I = (1, 3)$

Example

From Theorem (4), we deduce that the *IVP*

$$\begin{cases} 3y''' + 5y'' - y' + 7y = 0, \\ y(1) = 0, \;\; y'(1) = 0, \quad y''(1) = 0. \end{cases}$$

has a unique solution $y = 0$ on $\mathbb{R}$.

Definition (Linearly Dependent Solutions)

Let $f_1, \; f_2, \dots, f_n$ be n functions defined on an interval I. The functions $f_1, f_2, \dots, \; f_n$ are said to be linearly dependent on I if there exist n constants $c_1, c_2, \dots, c_n$ not all zero (i.e. $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$) such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \quad \text{for all} \quad x \in I.$$

Example

Prove that the functions

$$f_1(x) = x,\ \ f_2(x) = e^x,\ \ f_3(x) = xe^x,$$

and

$$f_4(x) = (2 - 3x)e^x,$$

are linearly dependent on $\mathbb{R}$.

Solution.

$$f_4(x) = (2 - 3x)e^x = 2e^x - 3xe^x = 2f_2(x) - 3f_3(x) + 0f_1(x),$$

hence

$$0f_1(x) + 2f_2(x) - 3f_3(x) - \ f_4(x) = 0,\ \text{ for all } \quad x \in \mathbb{R}.$$

So there exist $c_1 = 0$, $c_2 = 2$, $c_3 = -3$, and $c_4 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) = 0, \text{ for all } \quad x \in \mathbb{R}.$$

Then f_1, f_2, f_3 and f_4 are linearly dependent on $\mathbb{R}$.

Example

Show that $f_1(x) = \cos(2x)$, $f_2(x) = 1$, $f_3(x) = \cos^2(x)$ are linearly dependent on $\mathbb{R}$.

Solution.

We know that

$$f_3(x) = \cos^2(x) = \frac{1+\cos(2x)}{2} = \frac{1}{2}\, f_2(x) + \frac{1}{2}\, f_1(x),$$

for all $x \in \mathbb{R}$.

Then there exist $c_1 = c_2 = \frac{1}{2}$ and $c_3 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

So f_1, f_2, and f_3 are linearly dependent on $\mathbb{R}$.

Example

Show that

$$f_1(x) = 1,\ \ f_2(x) = \sec^2(x) \text{ and } \ f_3(x) = \tan^2(x)$$

are linearly dependent on $\left(0,\ \frac{\pi}{2}\right)$.

Solution. We know that

$$f_2(x) = \sec^2(x) = 1 + \tan^2(x) = f_1(x) + \ f_3(x),$$

hence

$$f_1(x) - \ f_2(x) + \ f_3(x) = 0 \ \text{ for all } x \in \left(0,\ \frac{\pi}{2}\right).$$

So there exist $c_1 = c_3 = 1$ and $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all} \quad x \in \left(0,\, \frac{\pi}{2}\right).$$

So $f_1,\ f_2$ and f_3 are linearly dependent on $x \in \left(0,\, \frac{\pi}{2}\right)$.

Definition (Linearly Independent Solutions)

Let $f_1, f_2, \ldots, f_n$ be n functions defined on an interval I. The functions $f_1, f_2, \ldots, f_n$ are said to be linearly independent on I if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0, \text{ for all } \quad x \in I.$$

is true only for $c_1 = c_2 = \ldots = c_n = 0$.

Example

Show that $f_1(x) = x$ and $f_2(x) = x^2$ are linearly independent on $I = [-1, 1]$.

Solution. Let $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } x \in I.$$

We have to prove that $c_1 = c_2 = 0$. As

$$c_1 x + c_2 x^2 = 0 \text{ for all } -1 \leq x \leq 1,$$

then for $x = 1$ and $x = -\frac{1}{2}$ we have

$$c_1 + c_2 = 0,$$

and

$$-\frac{1}{2}c_1 + \frac{1}{4}c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then f_1, and f_2 are linearly independent on I.

Example

Show that

$$f_1(x) = \sin(x)\, f_2(x) = \sin(2x),$$

are linearly independent on $I = [0, \pi)$.

Solution. Let $c_1, c_2 \in I$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \text{ for all } x \in I.$$

We have to show that $c_1 = c_2 = 0$. In fact for $x = \frac{\pi}{4}$, and $x = \frac{\pi}{3}$ we have

$$\begin{cases} c_1 \sin(\frac{\pi}{4}) + c_2 \sin(\frac{\pi}{2}) = 0, \\ c_1 \sin(\frac{\pi}{3}) + c_2 \sin(2\frac{\pi}{3}) = 0, \end{cases}$$

hence

$$\frac{1}{\sqrt{2}}c_1 + c_2 = 0, \quad \frac{\sqrt{3}}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then f_1, and f_2 are linearly independent on I.

Example

Show that

$$f_1(x) = 1, f_2(x) = e^x, \text{ and } \ f_3(x) = e^{-x}.$$

are linearly independent on $\mathbb{R}$.

Solution.

Let $c_1, \ c_2, \ c_3 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0, \quad \text{for all} \quad x \in \mathbb{R}.$$

We have to prove that $c_1 = c_2 = c_3 = 0$. In fact we have

$$c_1 + c_2 e^x + c_3 e^{-x} = 0, \quad \text{for all } x \in \mathbb{R},$$

then for the values $x = 0$, $x = 1$, $x = -1$, we have

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_{1+} c_2 e + c_3 e^{-1} = 0 \\ c_1 + c_2 e^{-1} + c_3 e = 0, \end{cases}$$

which implies that $c_1 = c_2 = c_3 = 0$. Then f_1, f_2 and f_3 are linearly independent on $\mathbb{R}$.

Now we shall obtain a sufficient condition that n functions are linearly independent on an interval I. Let us assume that each of the functions $f_1, f_2, \dots, f_n$ is differentiable at least $(n-1)$ times in the interval I. Let $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0, \text{ for all } x \in I. \qquad (5)$$

We have

$$\begin{cases} c_1 f_1'(x) + c_2 f_2'(x) + \ldots + c_n f_n'(x) = 0 \\ c_1 f_1''(x) + c_2 f_2''(x) + \ldots + c_n f_n''(x) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \ldots + c_n f_n^{(n-1)}(x) = 0, \end{cases}$$

for all $x \in I$. The nature of the solutions of these n linear equations in $c_1, c_2, \ldots, c_n$ will be determined by the value of the determinant

$$W(x, f_1, f_2 \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & \ldots & f_n'(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & \ldots & f_n^{(n-1)}(x). \end{vmatrix} \quad (6)$$

Now if $x_0 \in I$ such that $W(x_0, f_1, f_2, \ldots, f_n) \neq 0$, then $c_1 = c_2 = \ldots = c_n = 0$, and hence the functions $f_1, f_2, \ldots, f_n$ are linearly independent on I.

Definition

The function $W(x, f_1, f_2, \dots, f_n)$ defined by the equation (6) is called Wronskian of the functions f_1, f_2, $\dots$, f_n.

Example

Show that $f_1(x) = 1,\ f_2(x) = x,\ ,\dots,\ \ f_n(x) = x^{n-1}$ are linearly independent on $\mathbb{R}$.

Solution.

We calculate

$$W(x,\ f_1, f_2 \dots, f_n) = \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 0 & 1 & 2x & \dots & (n-1)x^{n-2} \\ 0 & 0 & 2 & \dots & (n-1)(n-2)x^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (n-1)! \end{vmatrix}$$

and we find $W(x,\ f_1,\ f_2,\ \dots,\ f_n) = 0!1!2!\dots(n-1)! \neq 0$ for all $x \in \mathbb{R}$. Then $f_1,\ f_2,\ \dots,\ f_n$ are linearly independent on $\mathbb{R}$.

Example

Prove that $f_1(x) = x^2,\ f_2(x) = x^2\ln(x)$ are linearly independent on $(0,\ \infty)$.

Solution.

We use the definition of

$$\begin{aligned} W(x,\ f_1, f_2) &= \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} \\ &= 2x^3\ln(x) + x^3 - 2x^3\ln(x) = x^3 \neq 0 \text{ for all } x \in (0,\infty)\,, \end{aligned}$$

then f_1 and f_2 are linearly independent on $(0,\ \infty)$.

Example

Show that

$$f_1(x) = x^2 \text{and } f_2(x) = x\,|x|\,,$$

are
(i) linearly dependent on [0, 1]
(ii) linearly independent on [−1, 1]

Solution.
(i) on [0,1] we have

$$f_1(x) = f_2(x) = x^2,$$

hence

$$f_1(x) - f_2(x) = 0, \text{ for all } 0 \leq x \leq 1.$$

So there exist $c_1 = 1$, $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \text{ for all } 0 \leq x \leq 1.$$

Then f_1 and f_2 are linearly dependent on [0, 1].
(ii) Let $c_1, c_2 \in \mathbb{R}$ be such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } -1 \leq x \leq 1,$$

hence

$$c_1 x^2 + c_2 x\,|x| = 0 \text{ for all} -1 \leq x \leq 1.$$

Now for $x = 1$ and $x = -1$ we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on $[-1, 1]$.

Remark 1 :

(**i**) If f_1, f_2, ..., f_n are linearly dependent on an interval I and each of the functions f_1, f_2,..., f_n is differentiable at least $(n-1)$ times on I, then

$$W(x, f_1, f_2, \dots, f_n) = 0, \text{ for all } x \in I.$$

For example, it was proved that

$$f_1(x) = 1,\ f_2(x) = \sec^2(x),\ \text{and}\ \ f_3(x) = \tan^2(x).$$

are linearly dependent on $\left(0,\ \frac{\pi}{2}\right)$, then

$$\begin{aligned} & W(x, f_1, f_2, f_3) \\ = & \begin{vmatrix} 1 & \sec^2(x) & \tan^2(x) \\ 0 & 2\sec^2(x)\tan(x) & 2\tan(x)\sec^2(x) \\ 0 & 4\sec^2(x)\tan^2(x) + 2\sec^4(x) & 4\sec^2(x)\tan^2(x) + 2\sec^4(x \end{vmatrix} \\ = & \ 0, \end{aligned}$$

for all $x \in \left(0,\ \frac{\pi}{2}\right)$.
(ii) If $W(x,\ f_1,\ f_2,\ \dots,\ f_n) = 0$ for all $x \in I$, then the functions f_1, f_2, $\dots$, f_n may be linearly independent or dependent on I.

Example

We consider the functions

$$f_1(x) = x^2 \text{ and } f_2(x) = x\,|x|\,.$$

on the interval $I = [-1, 1]$. Prove that

$$W(x, f_1, f_2) = 0, \quad \text{for all } x \in I.$$

Solution.

1. For $0 < x \leq 1$, we have

$$W(x,\ f_1, f_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0.$$

2. For $-1 \leq x < 0$, we have

$$W(x,\ f_1, f_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0.$$

3. For $x = 0$ we have

$$W(0, f_1(0), f_2(0)) = \begin{vmatrix} f_1(0) & f_2(0 \\ f_1'(0) & f_2'(0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

So $W(x, f_1, f_2) = 0$ for all $x \in [-1, 1]$, even these functions f_1 and f_2 are linearly independent on $[-1, 1]$ (see the example (13)), where $f_2'(0) = 0$.

The main result in this section is given by the following theorem.

Theorem

If $y_1, y_2, \ldots, y_n$ *are solutions of the differential equation*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0, \qquad (7)$$

where each $a_i(x)$ *is defined and continuous on an interval* I *and* $a_n(x) \neq 0$ *for all* $x \in I$*, then* $y_1, y_2, \ldots, y_n$ *are linearly independent on* I *if and only if*

$$W(x,\ y_1\, y_2, \ldots, y_n) \neq 0 \ \ \textit{for all} \ \ x \in I.$$

Example

We know that the functions x and x^2 are linearly independent on the interval $-1 \leq x \leq 1$. However

$$W(x,\ f_1(x),\ f_2(x)) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2,$$

so that

$$W(0,\ f_1(0), f_2\ (0)) = 0,\ \text{where}\ \ x = 0 \in I = [-1, -1]\,.$$

This fact does not contradict Theorem (22), because there is no second- order linear differential equation with the interval of definition $-1 \leq x \leq 1$ that has x and x^2 as solutions. We can verify that $y_1 = x$ and $y_2 = x^2$ are solutions of the second- order linear differential equation

$$x^2y'' - 2xy' + 2y = 0,$$

where the interval of definition I must exclude $x = 0$, since we have assumed that $a_2(x) = x^2 \neq 0$ in I. So that we conclude that the Theorem (4) is not contradicted by this example, and we should distinguish between the functions which are linearly independent on an interval I as algebraic functions, and the functions which are linearly independent on an interval I, and are solutions of a linear differential equation.

Example

It is easy to see that the functions

$$y_1 = x, \; y_2 = x^2,$$

and

$$y_3 = x^3.$$

are solutions of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0.$$

Show that y_1, y_2 and y_3 are linearly independent on $(0, \infty)$.

Solution.
Here we have $a_3(x) = x^3 \neq 0$ for all $x > 0$ or $x < 0$. By using the Wronskian we have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0.$$

for all $x \in (0, \infty)$, or for all $x \in (-\infty, 0)$. So y_1, y_2 and y_3 are linearly independent on $(0, \infty)$ or on $(-\infty, 0)$. But as algebraic functions y_1, y_2 and y_3 are linearly independent on $\mathbb{R}$.

Definition (Fundamental Set of Solutions)

Any set y_1, y_2, ..., y_n of n functions linearly independent solutions of the homogeneous linear nth-order differential equation (7) on an interval I is said to be a fundamental set of solutions on I.

Here the number of functions which form the fundamental set of solutions on i equals to the order of the equation (7).

Theorem

Let y_1, y_2, ..., y_n be a fundamental set of solutions of the homogeneous linear nth-order differential equation (7) on an interval I. Then for any solution y of Eq (7) on I, there exist n constants c_1, c_2,. $\ldots, c_n \in \mathbb{R}$, such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x). \tag{8}$$

Theorem (Existence of a fundamental set)

There exist a fundamental set of solutions for homogeneous linear nth-order differential equation (7) on an interval I.

Definition (General Solution of the Homogeneous Equation)

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of homogeneous linear nth-order differential equation (7) on an interval I.The general solution of the equation (7) on I is defined by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x), \quad x \in I,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. The general solution of (7) is also called the complete solution of (7).

Example

Verify that $y_1 = e^{2x}$, and $y_2 = e^{-3x}$ form a fundamental set of solutions of the differential equation

$$y'' + y' - 6y = 0,$$

and find the general solution.

Solution.
Substituting

$$y_1 = e^{2x}, y_1' = 2e^{2x},\ y_1'' = 4e^{2x},$$

in the differential equation, we get

$$4e^{2x} + 2e^{2x} - 6e^{2x} = 0.$$

Hence $y_1 = e^{2x}$, is a solution of the differential equation. By the same method we can prove that $y_2 = e^{-3x}$, is also a solution of the differential equation. We now have

$$W(x, e^{2x}, e^{-3x}) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -5e^{-x} \neq 0 \text{ for all } x \in \mathbb{R}.$$

Then y_1 and y_2 are linearly independent on $\mathbb{R}$. From Theorem (**??**), we deduce the general solution of the differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

where $c_1,\ c_2 \in \mathbb{R}$.

Example

It is easy to see that the functions

$$y_1 = e^x, \ y_2 = e^{2x}, \text{ and } y_3 = e^{3x},$$

are solutions of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

Find the general solution of the differential equation.

Solution.
Since

$$W(x, e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0,$$

for all $x \in \mathbb{R}$.

We deduce that

$$y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

is the general solution of the differential equation.

Example

Prove that

$$y_1 = x^3 e^x, \text{ and } y_2 = e^x,$$

are solutions of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0,$$

where $x > 0$. Find also the general solution of the differential equation.

Solution.

Substituting

$$y_1 = x^3 e^x, \; y_1' = 3x^2 e^x + x^3 e^x, \; y_1'' = 6xe^x + 6x^2 e^x + x^3 e^x,$$

in the differential equation we obtain

$$6x^2 e^x + 6x^3 e^x + x^4 e^x - 6x^3 e^x - 2x^4 e^{xe^x} - 6x^2 e^x + -2x^3 e^x + x^4 e^x + 2x^3 e^x = 0$$

Substituting

$$W(x, x^3e^x, e^x) = \begin{vmatrix} x^3e^x & e^x \\ 3x^2e^x + x^3e^x & e^x \end{vmatrix} = -3x^2e^x \neq 0, \text{ for all } \ x > 0.$$

Then

$$y_1 = x^3e^x,$$

and

$$y_2 = e^x.$$

are linearly independent on $(0, \infty)$, and we conclude that

$$y_c = c_1x^3e^x + c_2e^x,$$

is the general solution of the differential equation.

Remark 2 :
The property of general solution exists only in the homogeneous linear nth -order differential equation (7) but does not exist in the homogeneous non- linear differential equation, for example the differential equation

$$(xy' + 1)(yy' + 1) = 0.$$

is a non-linear first order differential equation has not general solution, because it has two family of curves of solutions $y = -\ln|xc_1|$ such that $x \neq 0$, and an arbitrary constant $c_1 \neq 0$, $y^2 + 2x = c_2$ where $y \neq 0$ and c_2 is an arbitrary constant.

Example

Given that

$$y = c_1 e^x + c_2 e^{-x},$$

is a two parameters family of solutions of

$$y'' - y = 0 \text{ on } (-\infty, \infty),$$

find a curve of the family satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.

Solution.

From Theorem (4) the initial value problem

$$\begin{cases} y''(x) - y(x) = 0 \\ y(0) = 0 \;\; y'(0) = 1, \end{cases}$$

has a unique solution.

For $y(0) = 0$ we have $c_1 + c_2 = 0$ and for $y'(0) = 1$ we have $c_1 - c_2 = 1$, hence $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. So the unique solution of the initial value problem is

$$y = \frac{1}{2}(e^x - e^{-x}) = \sinh(x).$$