Ordinary Differential Equations Systems With Constant Coefficients

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1 Elimination Method

Systems of ordinary differential equations arise in problems involving several dependent variables, each of which is a function of a single independent variable. We will use the following notations: $D = d/dt$, $D^2 = d^2/dt^2, \dots, D^n = d^n/dt^n$, where t is the independent variable. We may denote by y, u, v, w, z and so on for the dependent variables which are functions of t.
For example, the motion of a particle in space is governed by a system of the three following equations:

$$\begin{cases} mD^2u = f_1(t, u, v, w, D^1u, D^1v, D^1w) \\ mD^2v = f_2(t, u, v, w, D^1u, D^1v, D^1w) \\ mD^2w = f_3(t, u, v, w, D^1u, D^1v, D^1w), \end{cases}$$

where m is the mass of the particle, u, v, w are its spacial coordinates and f_1, f_2, f_3 are the forces acting on the particle the u, v and w directions respectively. We will use the method of elimination to solve linear systems of ordinary differential equations with constant coefficients. For systems of only two or three first order equations such method is quite efficient. It can be applied to nonhomogeneous systems as well as to homogeneous ones.

Elimination Method

The method of elimination can also be used to solve systems of higher order equations.

Consider the following system

$$\begin{cases} L_1[x] + L_2[y] = F_1(t), \\ L_3[x] + L_4[y] = F_2(t), \end{cases} \tag{1}$$

where L_1, L_2, L_3, and L_4 are linear differential operators with constant coefficients, and $F_1(t)$ and $F_2(t)$ are given functions. The system (1) is written in its operator form. The operators L_i, $i = 1, 2, 3, 4$ are commutative, that is

$$L_1 L_2[x] = L_2 L_1[x], L_1 L_3[x] = L_3 L_1[x], L_2 L_4[x] = L_4 L_2[x],$$

and so on. For example if

$$L_1[x] = (aD + b)[x]$$

and

$$L_2[x] = (cD + d)[x],$$

Remark 1 : The property of commutativity is possessed only by linear operators with constant coefficients and not by nonlinear ones.

Suppose now we want to solve System (1) by using the elimination method. To eliminate x we apply L_3 to first equation and L_1 to the second equation, we have

$$L_3 L_1[x] + L_3 L_2[y] = L_3[F_1],$$

$$L_1 L_3[x] + L_1 L_4[y] = L_1[F_2].$$

Then substrate the first from the second to obtain

$$L_1 L_4[y] - L_3 L_2[y] = L_1[F_2] - L_3[F_1]. \tag{4}$$

Equation (4) can be solved for y and then x can be found from either the first or second equation in (1).

Example

Find the general solution of the system

$$\begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} - 3\frac{dy}{dt} - x + 2y = 0, \\ \frac{dx}{dt} + 2\frac{dy}{dt} + 2x - 4y = 0. \end{cases} \tag{5}$$

Solution.

We first write the system (5) in its operator form

$$\begin{cases} (D^2+D-1)[x]+(D^2-3D+2)[y]=0, \\ (D+2)[x]+(2D-4)[y]=0. \end{cases} \tag{6}$$

To eliminate x, we apply the operator D^2+D-1 to the second equation in (6) and $D+2$ to the first one and substrate the first from the second, we get

$$\left((D^2+D-1)(2D-4)-(D+2)(D^2-3D+2)\right)[y]=0,$$

or

$$\left(D^3-D^2-2D\right)[y]=0 \Leftrightarrow y'''-y''-2y'=0. \tag{7}$$

The characteristic equation for Eq (7) is

$$m^3-m^2-2m=0$$

whose roots are $0, 2, -1$. Thus

$$y(t)=c_1+c_2e^{2t}+c_3e^{-t}.$$

Substitution of this last expression in the second equation of

Example

Solve the system

$$\begin{cases} x' = x - y + t, \\ y' = x + 3y - 3t. \end{cases} \tag{9}$$

Solution.

The system (4) can be written in the operator form

$$\begin{cases} (D-1)[x] + y = t, \\ (D-3)[y] - x = -3t. \end{cases} \tag{10}$$

To eliminate y, we apply $D-3$ to Eq (1) and substrate the second from the first, we obtain

$$\left(D^2 - 4D + 4\right)[x] = 1 \Leftrightarrow x'' - 4x' + 4x = 1. \tag{11}$$

The characteristic equation for the homogeneous equation in Eq (11) is

$$m^2 - 4m + 4 = 0,$$

which has the double root $m = 2$. Thus the general solution of the homogeneous equation

$$x'' - 4x' + 4x = 0$$

is

$$x_c(t) = (c_1 + c_2 t)e^{2t}.$$

We then use for example the method of undetermined coefficients to find that $x_p = 1/4$. thus the general solution of the nonhomogeneous (11) is given by

$$x(t) = (c_1 + c_2 t)e^{2t} + 1/4.$$

We deduce from the first equation in (9) that

$$y(t) = t + 1/4 - (c_1 + c_2 + c_2 t)e^{2t}.$$

Example

Solve the system

$$\left\{ \begin{array}{c} x'' + y' - 3x' + 2x - y = 0, \\ x' + y' - 2x + y = 0. \end{array} \right. \tag{12}$$

Solution.

We first write the system (12) in its operator form

$$\begin{cases} (D^2 - 3D + 2)[x] + (D-1)[y] = 0, \\ (D-2)[x] + (D+1)[y] = 0. \end{cases} \tag{13}$$

To eliminate y,we apply $(D+1)$ to the first equation in (13) and $(D-1)$ to the second and substract the first from the second, we get

$$\left(D^3 - 3D^2 + 2D\right)[x] = 0 \Leftrightarrow x''' - 3x'' + 2x' = 0. \tag{14}$$

The general solution of *Eq* (14) is

$$x(t) = c_1 + c_2 e^{2t} + c_3 e^t.$$

From *Eq* (2) in (12), we have

$$y' + y = c_3 e^t + 2c_1. \tag{15}$$

We solve the linear equation (15) to obtain

$$y(t) = 2c_1 + \frac{c_3}{2} e^t + c_4 e^{-t}.$$

Example

Solve the initial value problem

$$\begin{cases} x' + 5y - 2x = -\sin 2t, \quad x(0) = 0 \\ y' - x + 2y = t, \quad y(0) = 1, \end{cases} \tag{16}$$

Solution.
The system (16) has the operator form

$$\begin{cases} (D-2)[x] + 5y = -\sin 2t, \\ (D+2)[y] - x = t. \end{cases} \quad (17)$$

To eliminate x, we apply $(D-2)$ to the second equation and sum both equations, we obtain

$$(D^2-4)[y]+5y = 1-2t-\sin 2t \Leftrightarrow y''+y = 1-2t-\sin 2t. \quad (18)$$

To find the general solution of *Eq* (18), it is better to use the method of undetermined coefficients. We find that

$$y_c = c_1 \cos t + c_2 \sin t,$$

and

$$y_p = 1 - 2t + \frac{1}{3}\sin 2t.$$

Thus

$$y(t) = c_1 \cos t + c_2 \sin t + 1 - 2t + \frac{1}{3}\sin 2t.$$

From the second equation in (16), we have

$$x(t) = (2c_2 - c_1)\sin t + (c_2 + 2c_1)\cos t - 5t + \frac{2}{3}\sin 2t + \frac{2}{3}\cos 2t.$$

Initial conditions give $c_1 = 0$, and $c_2 = -\frac{2}{3}$.Then we get

$$x(t) = \frac{-4}{3}\sin t - \frac{2}{3}\cos t - 5t + \frac{2}{3}\sin 2t + \frac{2}{3}\cos 2t,$$

and

$$y(t) = \frac{-2}{3}\sin t + \frac{1}{3}\sin 2t + 1 - 2t.$$

Example

Find the general solution of the system

$$\begin{cases} \frac{1}{2}x''' - y'' = \cos t, \\ \frac{1}{2}x'' + x + y' = -\cos t. \end{cases} \tag{19}$$

Solution.

We write the system (19) in the operator form

$$\begin{cases} \frac{1}{2}D^3[x] - D^2[y] = \cos t, \\ (\frac{1}{2}D^2 + 1)[x] + D[y] = -\cos t. \end{cases}$$

To eliminate y, we apply the operator D to the second equation and then sum both equations

$$(D^3 + D)[x] = \sin t + \cos t \Leftrightarrow x''' + x' = \sin t + \cos t. \qquad (20)$$

By using the method of undetermined coefficients method, we find that the general solution of equation (20) is given by

$$x(t) = c_1 + c_2 \cos t + c_3 \sin t - \frac{t}{2}(\cos t + \sin t). \qquad (21)$$

Substitution of the expression (21) in the second equation of (19) gives

$$y(t) = (\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}) \cos t + (-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}) \sin t - c_1 t + c_4.$$

To eliminate the extra constant c_4, we substitute for $x(t)$ and $y(t)$ in the second equation in (19) and find that $c_4 = 0$. Hence

$$y(t) = (\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4})\cos t + (-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4})\sin t - c_1 t.$$

Example

Solve the initial value problem

$$\begin{cases} x'' - y = e^t, & x(0) = 0,\ y(0) = 0, \\ y'' - x = 0, & x'(0) = 0,\ y'(0) = 0. \end{cases} \tag{22}$$

Solution. We write the system (22) in its operator form

$$\begin{cases} D^2[x] - y = e^t \\ D^2[y] - x = 0. \end{cases} \tag{23}$$

To eliminate y, we operate by D^2 on the first differential equation in (22) and then sum with the second equation, we obtain

$$x^{(4)} - x = e^t.$$

We look for

$$x(t) = x_c(t) + x_p(t).$$

We solve the homogeneous equation

$$x^{(4)} - x = 0,$$

and find that

$$x_c(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

Then we use the undetermined coefficients method to find that

$$x_p(t) = \frac{t}{4}e^t.$$

Hence

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}e^t.$$

From the first equation in (23), we get

$$y(t) = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{t}{4}e^t - \frac{1}{2}e^t.$$

We now determine the constants c_i, $i = 1, 2, 3, 4$. The above initial conditions give the algebraic system

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 - c_3 - \frac{1}{2} = 0 \\ c_1 - c_2 + c_4 + \frac{1}{4} = 0 \\ c_1 - c_2 - c_4 - \frac{1}{4} = 0. \end{cases} \tag{24}$$

Solving the algebraic system (24), we obtain

$$c_1 = \frac{1}{8},\ c_2 = \frac{1}{8},\ c_3 = \frac{-1}{4},\ c_4 = -\frac{1}{4}.$$

Thus the solution of the system (22) is given by

$$\begin{aligned} x(t) &= \frac{1}{8}e^t + \frac{1}{8}e^{-t} - \frac{1}{4}\cos t - \frac{1}{4}\sin t + \frac{t}{4}e^t, \\ y(t) &= \frac{1}{8}e^t + \frac{1}{8}e^{-t} + \frac{1}{4}\sin t + \frac{1}{4}\cos t + \frac{t}{4}e^t - \frac{1}{2}e^t. \end{aligned}$$

Example

Find the general solution of the system

$$\begin{cases} x'' - y' + x + y = 0 \\ y'' + x' + x - y = 0. \end{cases} \tag{25}$$

Solution.

The system (16) has the operator form

$$\begin{cases} (D^2+1)[x]+(1-D)[y]=0 \\ (D+1)[x]+(D^2-1)[y]=0. \end{cases} \tag{26}$$

To eliminate y, we apply the operator $D+1$ to the first equation in (26) and add the resulted equation to the second equation, we obtain

$$(D+1)\,(D^2+1)[x]+(D+1)[x]=0 \Leftrightarrow x'''+x''+2x'+2x=0. \tag{27}$$

The general solution of *Eq* (27) is given by

$$x(t)=c_1e^{-t}+c_2\cos\sqrt{2}t+c_3\sin\sqrt{2}t.$$

From the first equation in (25), we have the equation

$$y'-y=2c_1e^{-t}-c_2\cos\sqrt{2}t-c_3\sin\sqrt{2}t. \tag{28}$$

The differential equation (28) is a linear equation, we solve it and find that

$$y(t) = -c_1 e^{-t} + \frac{c_2}{3}\cos\sqrt{2}t - \frac{c_2\sqrt{2}}{3}\sin\sqrt{2}t + \frac{c_3}{3}\sin\sqrt{2}t + \frac{c_3\sqrt{2}}{3}\cos\sqrt{2}t +$$

In the same way as before we find $c_4 = 0$. Hence

$$y(t) = -c_1 e^{-t} + \frac{c_2}{3}\cos\sqrt{2}t - \frac{c_2\sqrt{2}}{3}\sin\sqrt{2}t + \frac{c_3}{3}\sin\sqrt{2}t + \frac{c_3\sqrt{2}}{3}\cos\sqrt{2}t.$$

Example

Solve the system

$$\begin{cases} y^{(4)} - x'' + \frac{1}{2}y^{(3)} = 1, \\ y' + 2x = -t^2. \end{cases} \tag{29}$$

Solution.

If we write the given system (29) in the operator form, we have

$$\begin{cases} (D^4 + D^3/2)[y] - D^2[x] = 1 \\ D[y] + 2x = -t^2. \end{cases} \tag{30}$$

To eliminate x, we multiply the first equation by (2) and operate by D^2 on the second equation and sum the two obtained equations, we obtain

$$(2D^4 + 2D^3)[y] = 0 \Leftrightarrow y^{(4)} + y^{(3)} = 0. \tag{31}$$

Eq (31) has the general solution

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}. \tag{32}$$

We infer from the second equation in (30) and (32) that

$$x(t) = \frac{-c_2}{2} - c_3 t + \frac{c_4}{2} e^{-t} - \frac{t^2}{2}.$$

Example

Solve the initial value problem

$$\begin{cases} x'' - y'' + x' + y' - x - y = e^t \\ y' - x = 0 \\ x(0) = 0,\ y(0) = 0,\ x'(0) = 0. \end{cases} \tag{33}$$

Solution.

The system (33) has the form

$$\begin{cases} (D^2 + D - 1)[x] + (-D^2 + D - 1)[y] = e^t \\ D[y] - x = 0. \end{cases} \tag{34}$$

To eliminate x, we apply the differential operator $D^2 + D - 1$ to the second equation and sum with the second equation, we obtain

$$(D^3 - 1)[y] = e^t \Leftrightarrow y^{(3)} - y = e^t, \tag{35}$$

Solving *Eq* (35), we obtain

$$y(t) = c_1 e^t + (c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t)e^{-t/2}. \tag{36}$$

The second equation in (33) and (36) give

$$x(t) = c_1 e^t + ((\sqrt{3}c_3 - \frac{c_2}{2})\cos\sqrt{3}t - (\sqrt{3}c_3 + \frac{c_2}{2})\sin\sqrt{3}t)e^{-t/2}.$$

Using initial conditions, we solve the algebraic system

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + \sqrt{3}c_3 - \frac{c_2}{2} = 1 \\ c_1 + (\frac{1}{4} - \frac{\sqrt{3}}{2})c_2 + (-3 - \frac{\sqrt{3}}{2})c_3 = 0, \end{cases}$$

and find that

$$c_1 = \frac{18 + \sqrt{3}}{36 + 6\sqrt{3}}, c_2 = -\frac{18 + \sqrt{3}}{36 + 6\sqrt{3}}, c_3 = \frac{3 + 2\sqrt{3}}{24 + 4\sqrt{3}}.$$

Hence, the solution of the (IVP) (33) is given by

$$
\begin{aligned}
x(t) &= \frac{18+\sqrt{3}}{36+6\sqrt{3}}e^t + \\
&+ \left(\left(\frac{18+\sqrt{3}}{36+6\sqrt{3}} + \frac{18+\sqrt{3}}{72+12\sqrt{3}}\right)\cos\sqrt{3}t \right. \\
&\left. -\left(\frac{18+\sqrt{3}}{36+6\sqrt{3}} + \frac{18+\sqrt{3}}{72+12\sqrt{3}}\right)\sin\sqrt{3}t \right) e^{-t/2}. \\
y(t) &= \frac{18+\sqrt{3}}{36+6\sqrt{3}}e^t + \left(-\frac{18+\sqrt{3}}{36+6\sqrt{3}}\cos\sqrt{3}t + \frac{3+2\sqrt{3}}{24+4\sqrt{3}}\sin\sqrt{3}t \right) e^{-}
\end{aligned}
$$

Example

Solve the initial value problem

$$\begin{cases} x^{(3)} - y = 0, \\ y' - x = \frac{1}{8}e^{2t} - \frac{1}{8}\sin 2t. \end{cases} \tag{37}$$

Solution.

We first write the system (37) in its operator form

$$\begin{cases} D^3[x] - y = 0 \\ D[y] - x = \frac{1}{8}e^{2t} - \frac{1}{8}\sin 2t. \end{cases} \tag{38}$$

In order to eliminate the unknown x, we apply the operator D^3 to the second equation in (38) and sum with the first one, we get

$$(D^4 - 1)[y] = e^{2t} + \cos 2t \Leftrightarrow y^{(4)} - y = e^{2t} + \cos 2t. \tag{39}$$

Equation (39) has the general solution

$$y(t) = y_p(t) + y_c(t),$$

that is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{15}e^{2t} + \frac{1}{15}\cos 2t. \tag{40}$$

It follows from the second equation in (37) and (40) that

$$x(t) = c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t + \frac{1}{120}e^{2t} - \frac{1}{120^{‘}}\sin 2t.$$