# Chapter 10

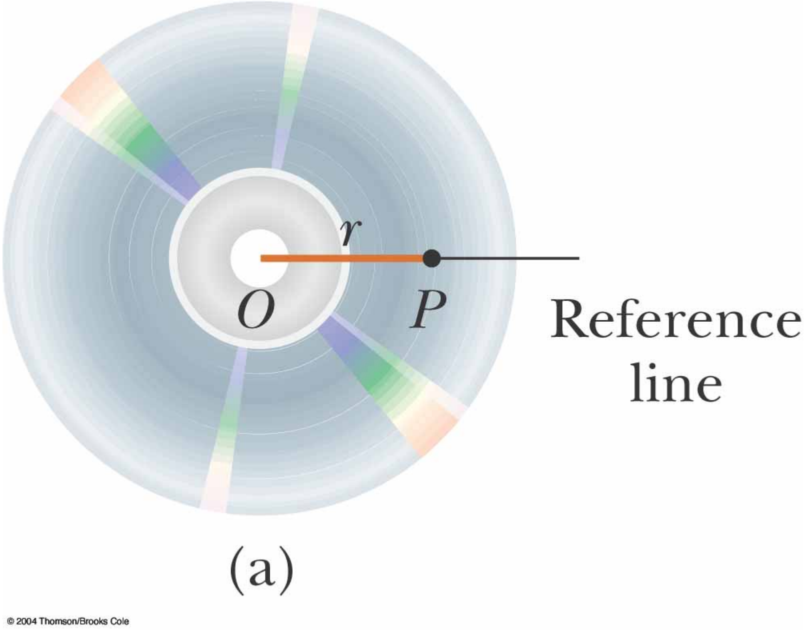
Rotation of a Rigid Object about a Fixed Axis

# Rigid Object

- A rigid object is one that is nondeformable
  - The relative locations of all particles making up the object remain constant
  - All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible

# Angular Position

- Axis of rotation is the center of the disc
- Choose a fixed reference line
- Point *P* is at a fixed distance *r* from the origin

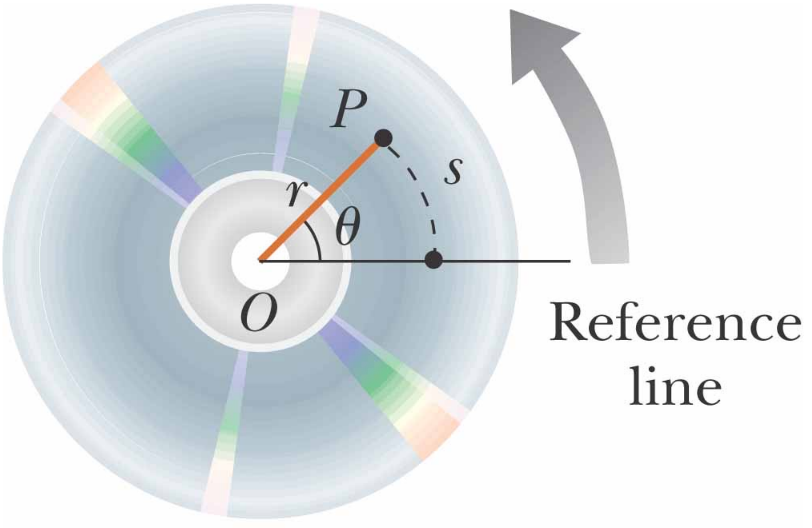


# Angular Position, 2

- Point *P* will rotate about the origin in a circle of radius *r*
- ***Every*** particle on the disc undergoes circular motion about the origin, *O*
- Polar coordinates are convenient to use to represent the position of *P* (or any other point)
- *P* is located at ($r$, $\theta$) where *r* is the distance from the origin to *P* and $\theta$ is the measured counterclockwise from the reference line

# Angular Position, 3

- As the particle moves, the only coordinate that changes is $\theta$
- As the particle moves through $\theta$, it moves though an arc length *s*.
- The arc length and *r* are related:
  - $s = \theta r$




(b)

# Radian

- This can also be expressed as

$$\theta = \frac{s}{r}$$

- $\theta$ is a pure number, but commonly is given the artificial unit, radian
- ***One radian is the angle subtended by an arc length equal to the radius of the arc***

# Conversions

- Comparing degrees and radians

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

- Converting from degrees to radians

$$\theta\,[\text{rad}] = \frac{\pi}{180^\circ}\theta \; [\text{degrees}]$$

# Angular Position, final

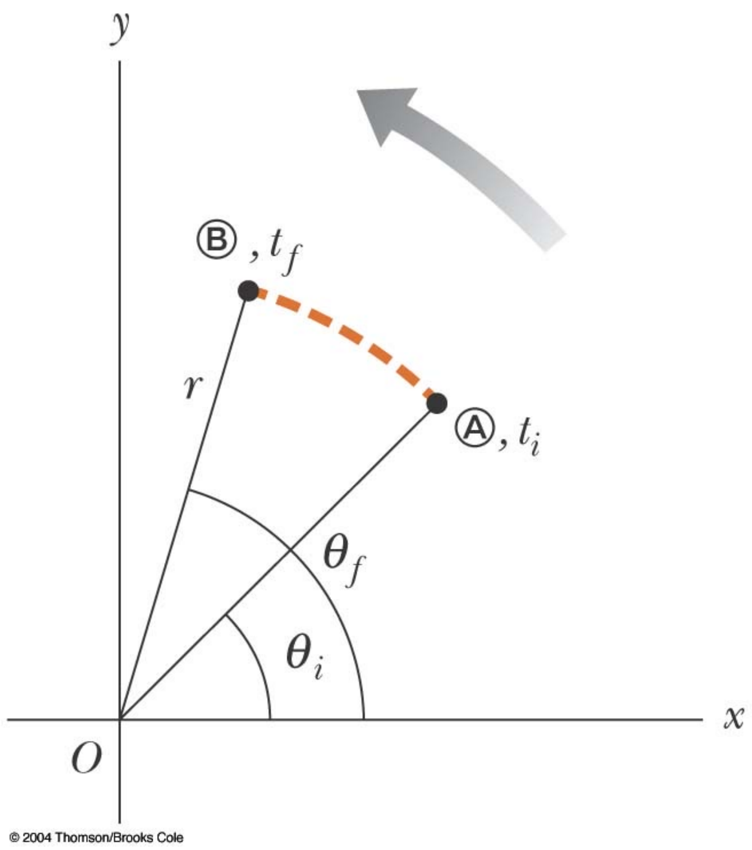
- We can associate the angle $\theta$ with the entire rigid object as well as with an individual particle
  - Remember every particle on the object rotates through the same angle
- The ***angular position*** of the rigid object is the angle $\theta$ between the reference line on the object and the fixed reference line in space
  - The fixed reference line in space is often the *x*-axis

# Angular Displacement

- The *angular displacement* is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- This is the angle that the reference line of length *r* sweeps out



# Average Angular Speed

- The average angular speed, $\omega$, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

# Instantaneous Angular Speed

- The *instantaneous* angular speed is defined as the limit of the average speed as the time interval approaches zero

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

# Angular Speed, final

- Units of angular speed are radians/sec
  - rad/s or $s^{-1}$ since radians have no dimensions
- Angular speed will be positive if $\theta$ is increasing (counterclockwise)
- Angular speed will be negative if $\theta$ is decreasing (clockwise)

# Average Angular Acceleration

- The average angular acceleration, $\alpha$, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

# Instantaneous Angular Acceleration

- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

# Angular Acceleration, final

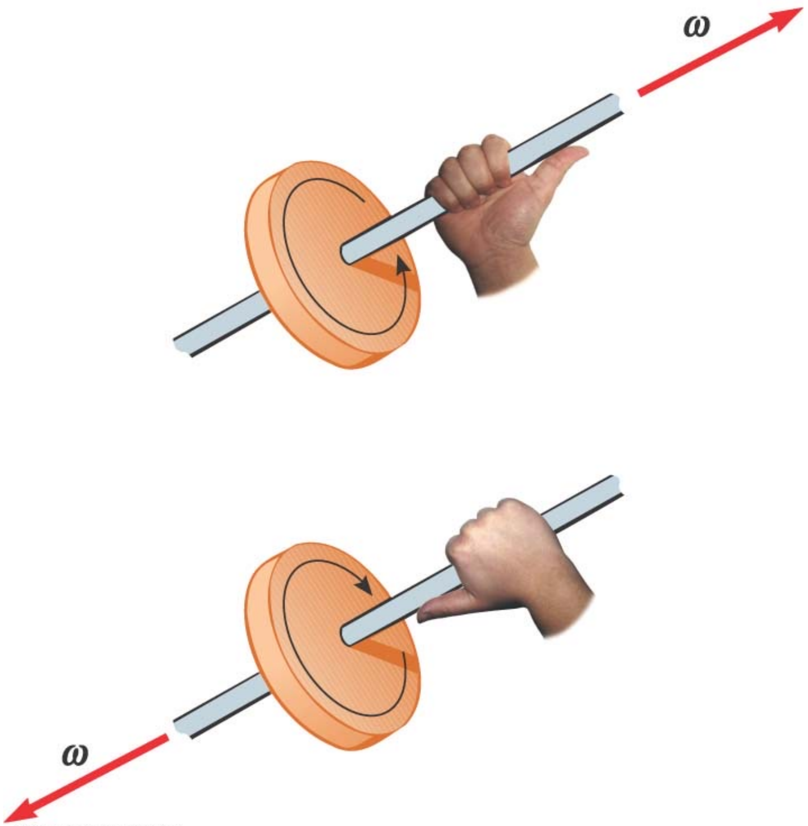
- Units of angular acceleration are rad/s² or $s^{-2}$ since radians have no dimensions
- Angular acceleration will be positive if an object rotating counterclockwise is speeding up
- Angular acceleration will also be positive if an object rotating clockwise is slowing down

# Angular Motion, General Notes

- When a rigid object rotates about a fixed axis in a given time interval, every portion on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration
  - So $\theta$, $\omega$, $\alpha$ all characterize the motion of the entire rigid object as well as the individual particles in the object

# Directions, details

- Strictly speaking, the speed and acceleration ($\omega$, $\alpha$) are the magnitudes of the velocity and acceleration vectors
- The directions are actually given by the right-hand rule



# Hints for Problem-Solving

- Similar to the techniques used in linear motion problems
  - With constant angular acceleration, the techniques are much like those with constant linear acceleration
- There are some differences to keep in mind
  - For rotational motion, define a rotational axis
    - The choice is arbitrary
    - Once you make the choice, it must be maintained
  - The object keeps returning to its original orientation, so you can find the number of revolutions made by the body

# Rotational Kinematics

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
  - These are similar to the kinematic equations for linear motion
  - The rotational equations have the same mathematical form as the linear equations

# Rotational Kinematic Equations

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

$$\theta_f = \theta_i + \frac{1}{2}\left(\omega_i + \omega_f\right)t$$

# Comparison Between Rotational and Linear Equations

**Table 10.1**

**Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration**

| Rotational Motion About Fixed Axis | Linear Motion |
|---|---|
| $\omega_f = \omega_i + \alpha t$ | $v_f = v_i + at$ |
| $\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$ | $x_f = x_i + v_i t + \frac{1}{2}at^2$ |
| $\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$ | $v_f^2 = v_i^2 + 2a(x_f - x_i)$ |
| $\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$ | $x_f = x_i + \frac{1}{2}(v_i + v_f)t$ |

# Relationship Between Angular and Linear Quantities

- Displacements

  $$s = \theta r$$

- Speeds

  $$v = \omega r$$

- Accelerations

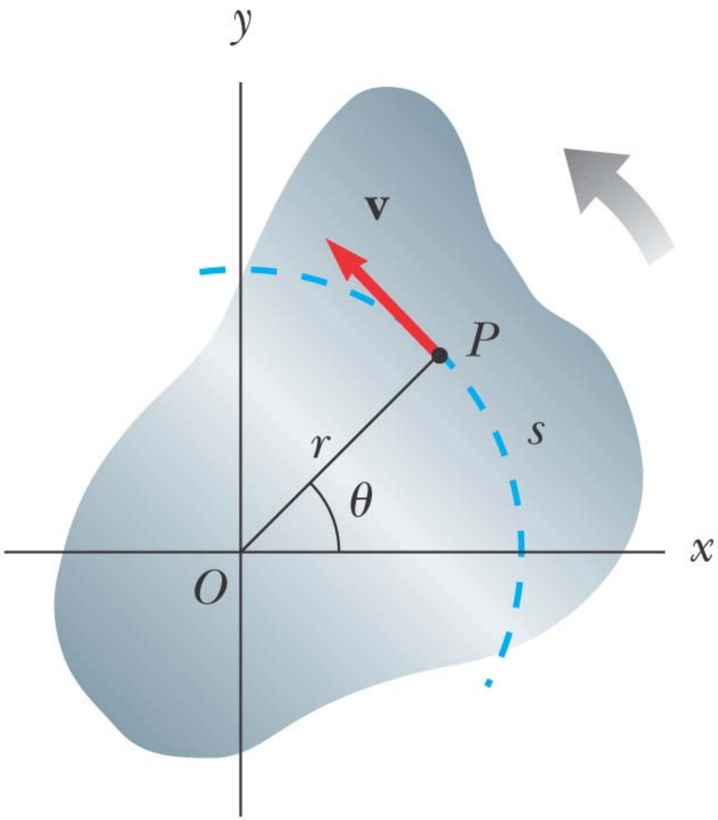
  $$a = \alpha r$$

- Every point on the rotating object has the same angular motion
- Every point on the rotating object does *not* have the same linear motion

# Speed Comparison

- The linear velocity is always tangent to the circular path
  - called the tangential velocity
- The magnitude is defined by the tangential speed

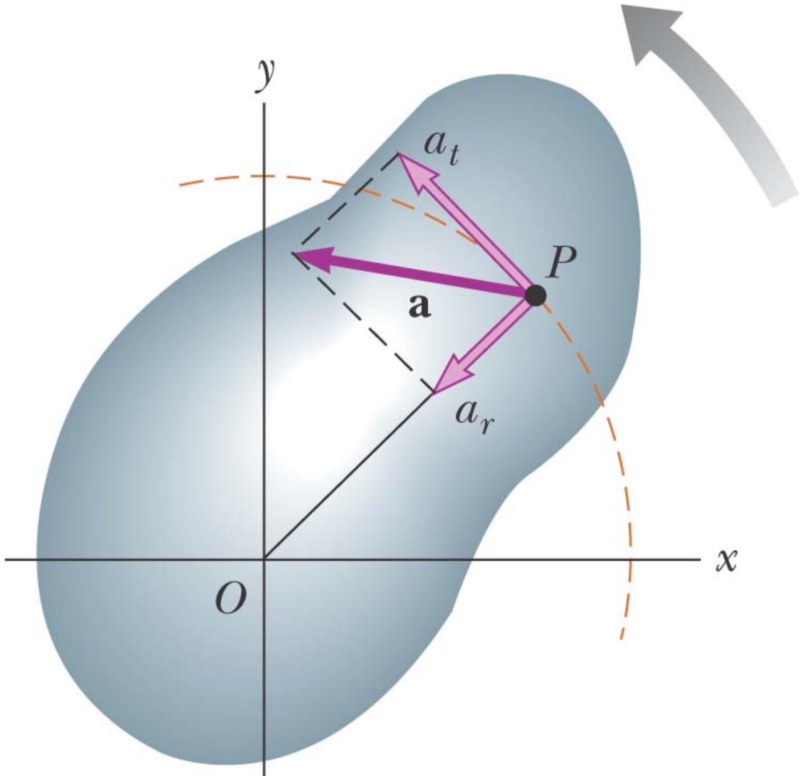
$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$



# Acceleration Comparison

- The tangential acceleration is the derivative of the tangential velocity

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$



# Speed and Acceleration Note

- All points on the rigid object will have the same *angular speed*, but **not** the same *tangential speed*
- All points on the rigid object will have the same *angular acceleration*, but **not** the same *tangential acceleration*
- The tangential quantities depend on *r*, and *r* is not the same for all points on the object

# Centripetal Acceleration

- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
  - Therefore, each point on a rotating rigid object will experience a centripetal acceleration

$$a_C = \frac{v^2}{r} = r\omega^2$$

# Resultant Acceleration

- The tangential component of the acceleration is due to changing speed
- The centripetal component of the acceleration is due to changing direction
- Total acceleration can be found from these components

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

# Rotational Motion Example

- For a compact disc player to read a CD, the angular speed must vary to keep the tangential speed constant ($v_t = \omega r$)
- At the inner sections, the angular speed is faster than at the outer sections

# Rotational Kinetic Energy

- An object rotating about some axis with an angular speed, $\omega$, has rotational kinetic energy even though it may not have any translational kinetic energy
- Each particle has a kinetic energy of
  - $K_i = ½ \, m_i v_i^2$
- Since the tangential velocity depends on the distance, $r$, from the axis of rotation, we can substitute $v_i = \omega_i \, r$

# Rotational Kinetic Energy, cont

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2$$

- Where $I$ is called the moment of inertia

# Rotational Kinetic Energy, final

- There is an analogy between the kinetic energies associated with linear motion ($K = \frac{1}{2} mv^2$) and the kinetic energy associated with rotational motion ($K_R = \frac{1}{2} I\omega^2$)
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object
- The units of rotational kinetic energy are Joules (J)

# Moment of Inertia

- The definition of moment of inertia is
  $$I = \sum_i r_i^2 m_i$$
- The dimensions of moment of inertia are $ML^2$ and its SI units are $kg \cdot m^2$
- We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass $\Delta m_i$

# Moment of Inertia, cont

- We can rewrite the expression for $I$ in terms of $\Delta m$

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

- With the small volume segment assumption,

$$I = \int \rho r^2 dV$$

- If $\rho$ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known

# Notes on Various Densities

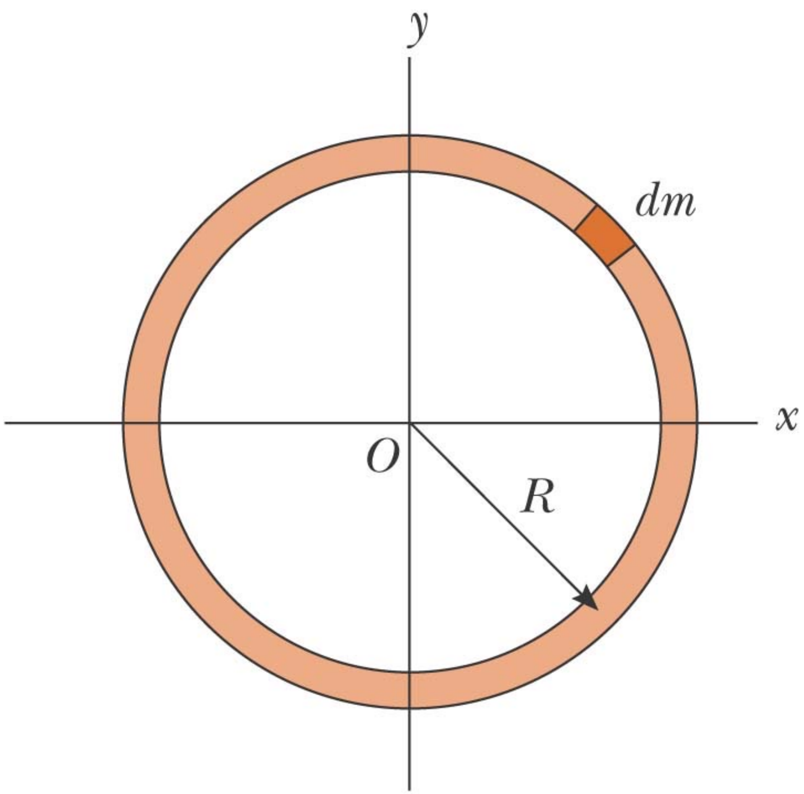
- Volumetric Mass Density –> mass per unit volume: $\rho = m / V$
- Face Mass Density –> mass per unit thickness of a sheet of uniform thickness, $t$: $\sigma = \rho t$
- Linear Mass Density –> mass per unit length of a rod of uniform cross-sectional area: $\lambda = m / L = \rho A$

# Moment of Inertia of a Uniform Thin Hoop

- Since this is a thin hoop, all mass elements are the same distance from the center

$$I = \int r^2 dm = R^2 \int dm$$

$$I = MR^2$$

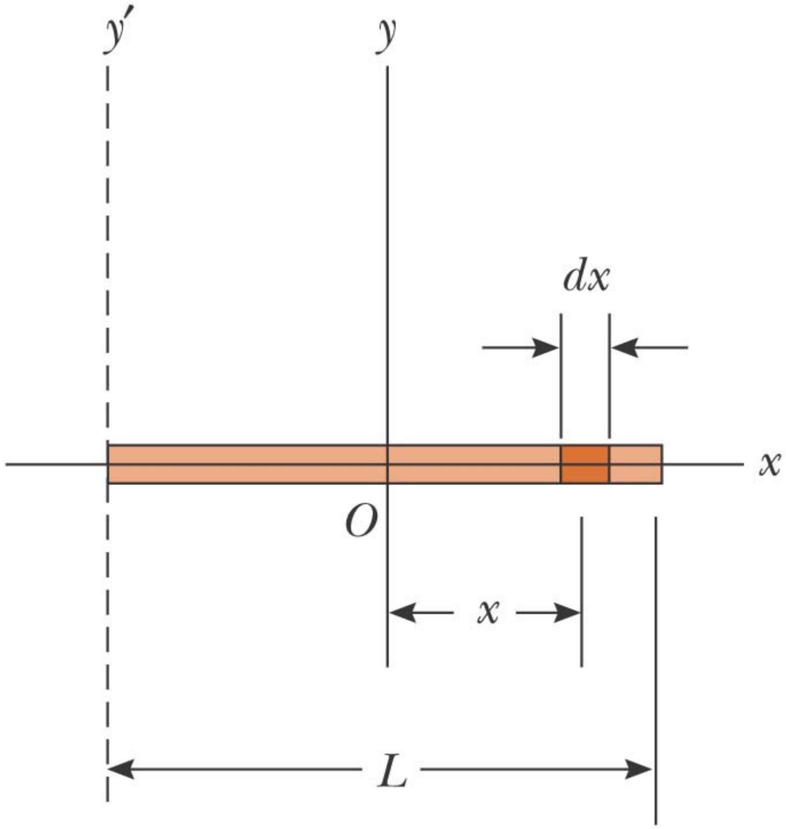


# Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass
  - $dm = \lambda\, dx$
- Then the moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx$$

$$I = \frac{1}{12} ML^2$$

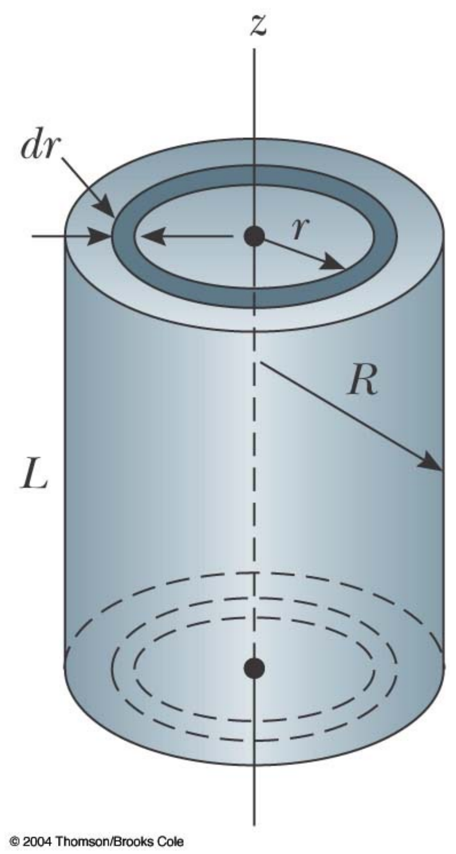


# Moment of Inertia of a Uniform Solid Cylinder

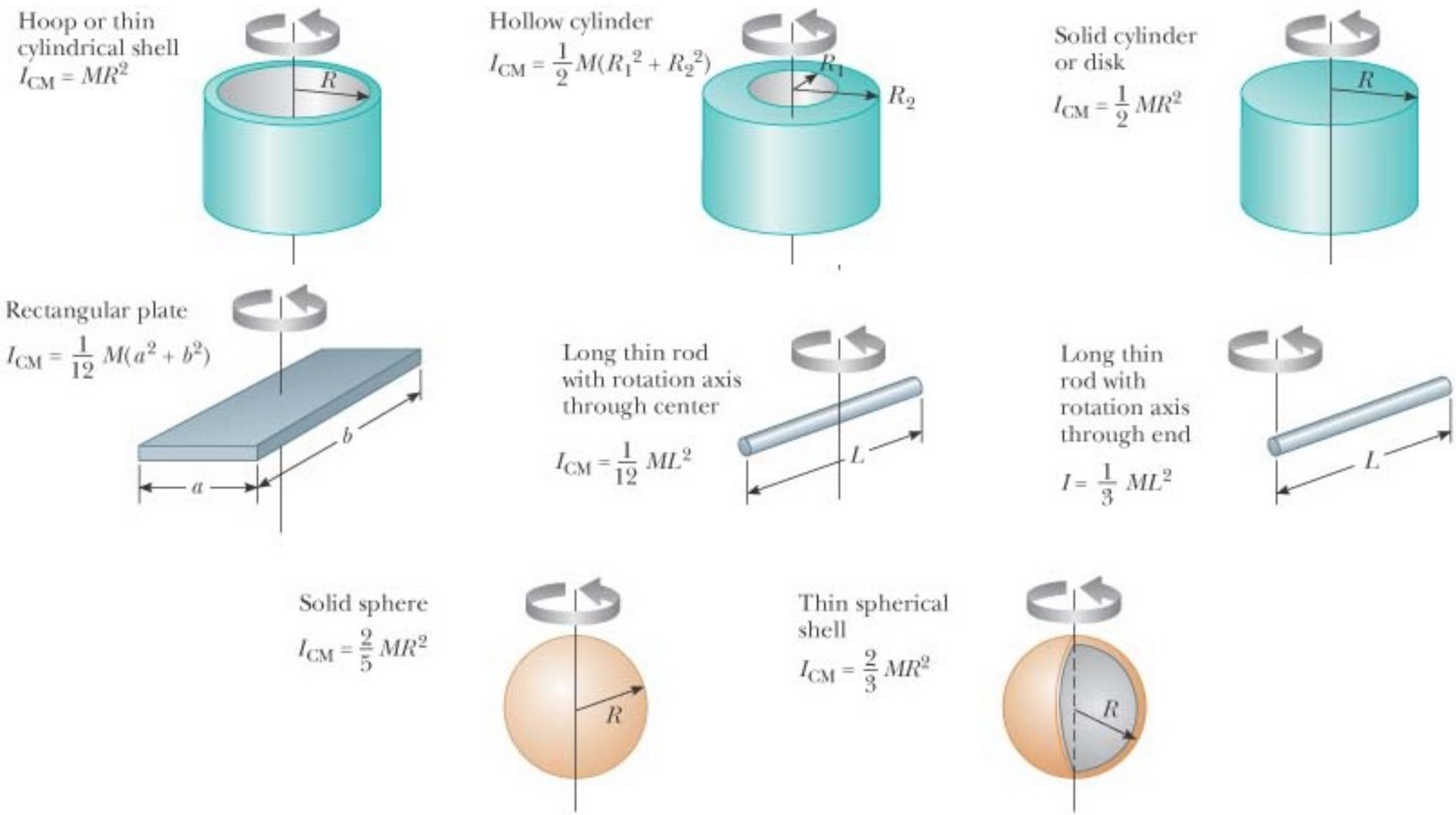
- Divide the cylinder into concentric shells with radius *r*, thickness *dr* and length *L*
- Then for *I*

$$I = \int r^2 dm = \int r^2 \left(2\pi\rho L r \, dr\right)$$

$$I_z = \frac{1}{2} MR^2$$



# Moments of Inertia of Various Rigid Objects

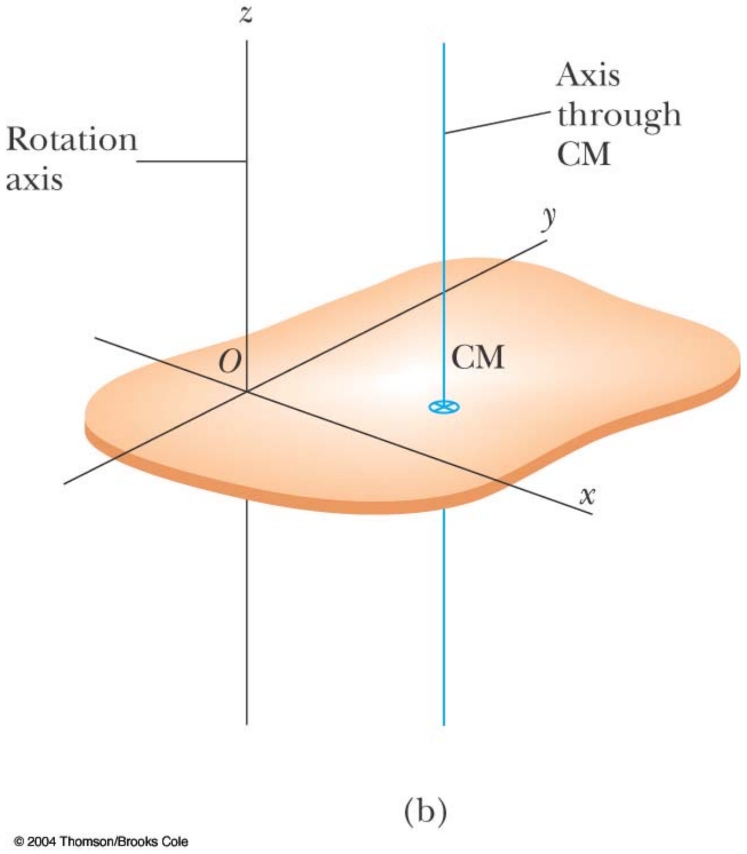


# Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem often simplifies calculations
- The theorem states $I = I_{CM} + MD^2$
  - $I$ is about any axis parallel to the axis through the center of mass of the object
  - $I_{CM}$ is about the axis through the center of mass
  - $D$ is the distance from the center of mass axis to the arbitrary axis

# Parallel-Axis Theorem Example

- The axis of rotation goes through $O$
- The axis through the center of mass is shown
- The moment of inertia about the axis through $O$ would be $I_O = I_{CM} + MD^2$



# Moment of Inertia for a Rod Rotating Around One End

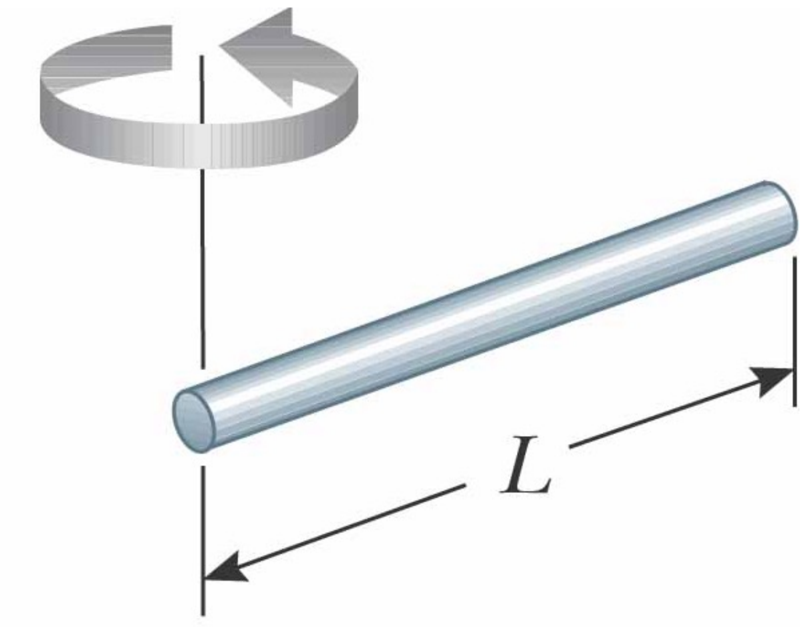
- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12} ML^2$$

- *D* is ½ *L*
- Therefore,

$$I = I_{\mathrm{CM}} + MD^2$$

$$I = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$$

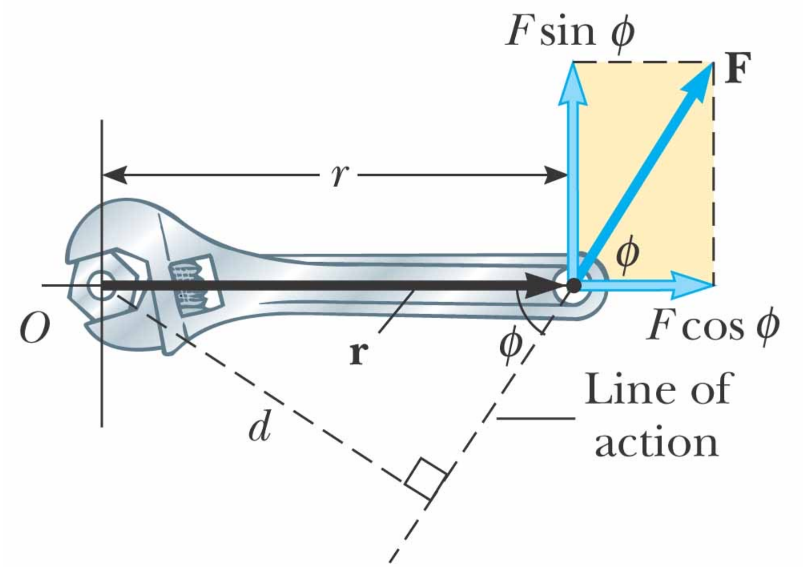


# Torque

- Torque, $\tau$, is the tendency of a force to rotate an object about some axis
  - Torque is a vector
  - $\tau = r F \sin \phi = F d$
    - **F** is the force
    - $\phi$ is the angle the force makes with the horizontal
    - $d$ is the *moment arm* (or lever arm)

# Torque, cont

- The moment arm, *d*, is the *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force
  - $d = r \sin \Phi$

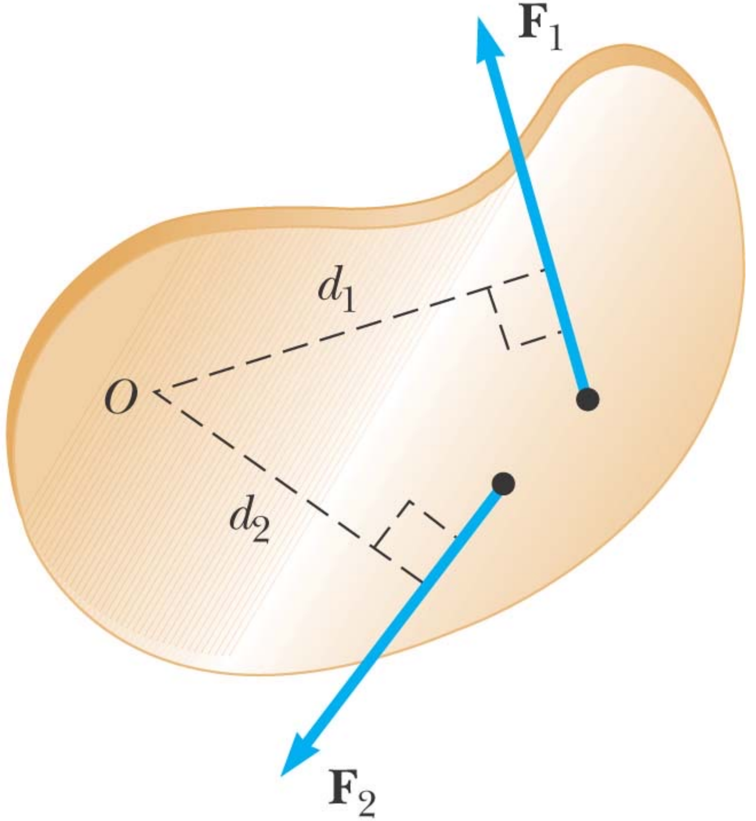


# Torque, final

- The horizontal component of **F** ($F \cos \phi$) has no tendency to produce a rotation
- Torque will have direction
  - If the turning tendency of the force is counterclockwise, the torque will be positive
  - If the turning tendency is clockwise, the torque will be negative

# Net Torque

- The force $\mathbf{F}_1$ will tend to cause a counterclockwise rotation about *O*
- The force $\mathbf{F}_2$ will tend to cause a clockwise rotation about *O*
- $\Sigma\tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$



# Torque vs. Force

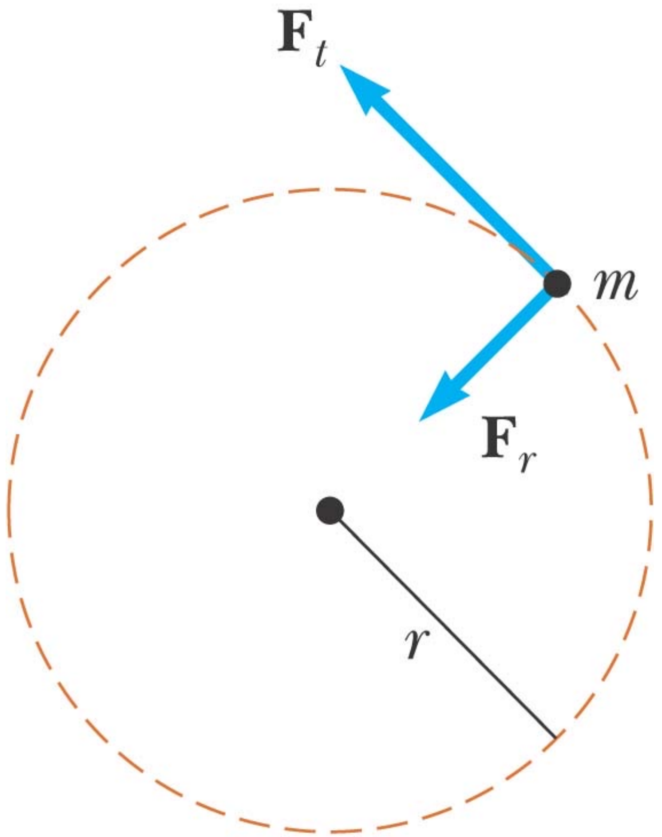
- Forces can cause a change in linear motion
  - Described by Newton's Second Law
- Forces can cause a change in rotational motion
  - The effectiveness of this change depends on the force *and* the moment arm
  - The change in rotational motion depends on the torque

# Torque Units

- The SI units of torque are N·m
  - Although torque is a force multiplied by a distance, it is very different from work and energy
  - The units for torque are reported in N·m and not changed to Joules

# Torque and Angular Acceleration

- Consider a particle of mass *m* rotating in a circle of radius *r* under the influence of tangential force $\mathbf{F}_t$
- The tangential force provides a tangential acceleration:
  - $F_t = ma_t$

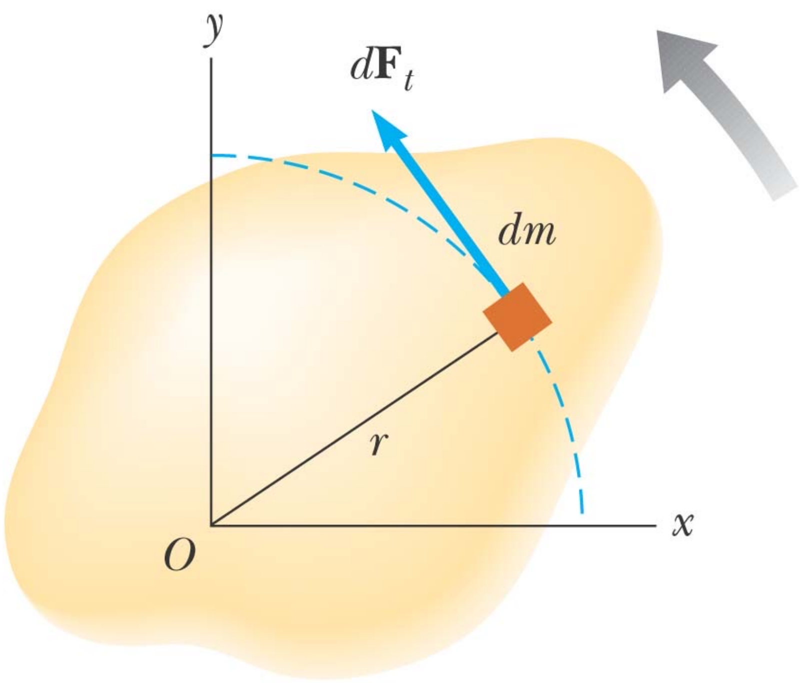


# Torque and Angular Acceleration, Particle cont.

- The magnitude of the torque produced by $\mathbf{F}_t$ around the center of the circle is
  - $\tau = F_t\, r = (ma_t)\, r$
- The tangential acceleration is related to the angular acceleration
  - $\tau = (ma_t)\, r = (mr\alpha)\, r = (mr^2)\, \alpha$
- Since $mr^2$ is the moment of inertia of the particle,
  - $\tau = I\alpha$
  - The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia

# Torque and Angular Acceleration, Extended

- Consider the object consists of an infinite number of mass elements *dm* of infinitesimal size
- Each mass element rotates in a circle about the origin, *O*
- Each mass element has a tangential acceleration



# Torque and Angular Acceleration, Extended cont.

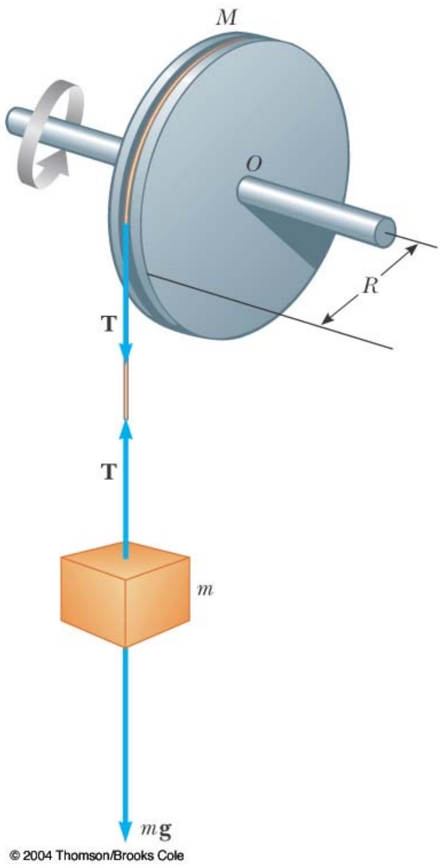
- From Newton's Second Law
  - $dF_t = (dm)\, a_t$
- The torque associated with the force and using the angular acceleration gives
  - $d\tau = r\, dF_t = a_t r\, dm = \alpha r^2\, dm$
- Finding the net torque
  - $\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm$
  - This becomes $\Sigma\tau = I\alpha$

# Torque and Angular Acceleration, Extended final

- This is the same relationship that applied to a particle
- The result also applies when the forces have radial components
  - The line of action of the radial component must pass through the axis of rotation
  - These components will produce zero torque about the axis

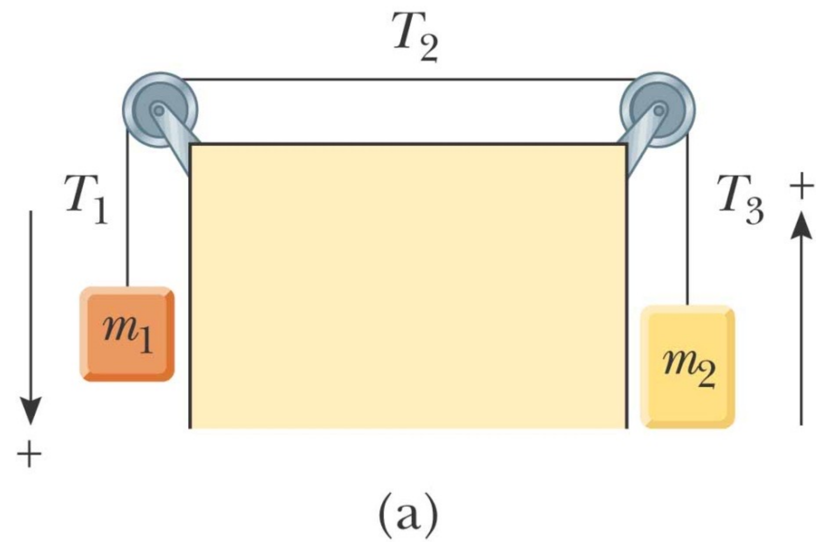
# Torque and Angular Acceleration, Wheel Example

- The wheel is rotating and so we apply $\Sigma\tau = I\alpha$
  - The tension supplies the tangential force
- The mass is moving in a straight line, so apply Newton's Second Law
  - $\Sigma F_y = ma_y = mg - T$

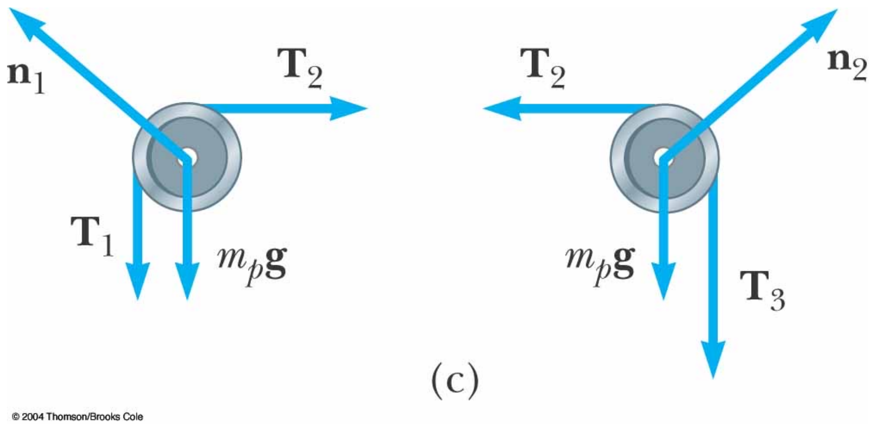


# Torque and Angular Acceleration, Multi-body Ex., 1

- Both masses move in linear directions, so apply Newton's Second Law
- Both pulleys rotate, so apply the torque equation



# Torque and Angular Acceleration, Multi-body Ex., 2




- The *m***g** and **n** forces on each pulley act at the axis of rotation and so supply no torque
- Apply the appropriate signs for clockwise and counterclockwise rotations in the torque equations

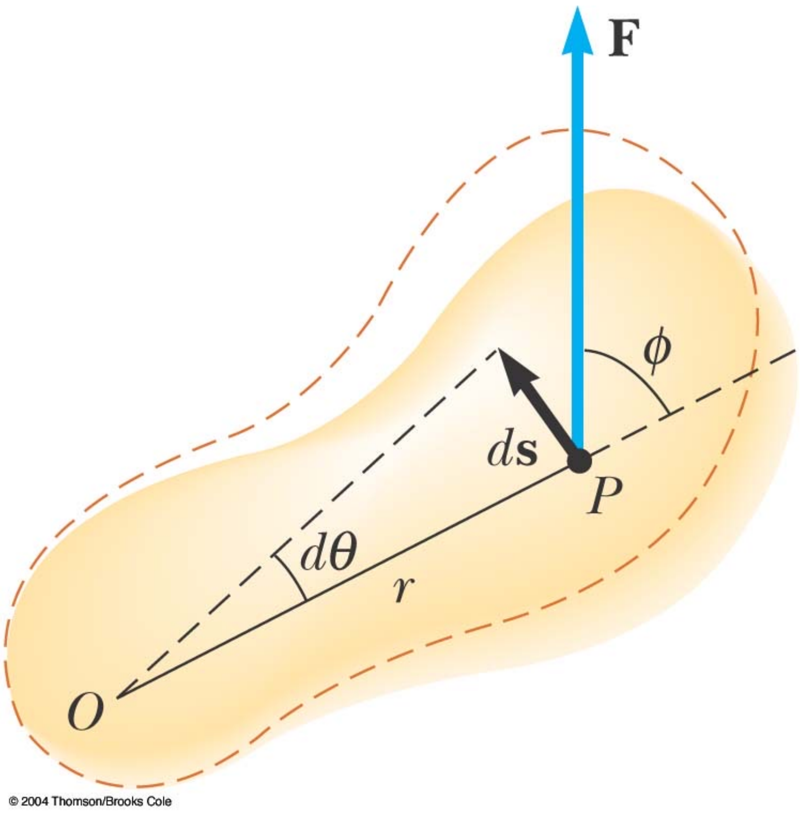
# Work in Rotational Motion

- Find the work done by **F** on the object as it rotates through an infinitesimal distance $ds = r\,d\theta$
- $dW = \mathbf{F} \cdot d\,\mathbf{s}$

  $= (F \sin \phi)\, r\, d\theta$

  $dW = \tau\, d\theta$

  The radial component of **F** does no work because it is perpendicular to the displacement



# Power in Rotational Motion

- The rate at which work is being done in a time interval *dt* is

$$\text{Power} = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

- This is analogous to $P = Fv$ in a linear system

# Work-Kinetic Energy Theorem in Rotational Motion

- The work-kinetic energy theorem for rotational motion states that *the net work done by external forces in rotating a symmetrical rigid object about a fixed axis equals the change in the object's rotational kinetic energy*

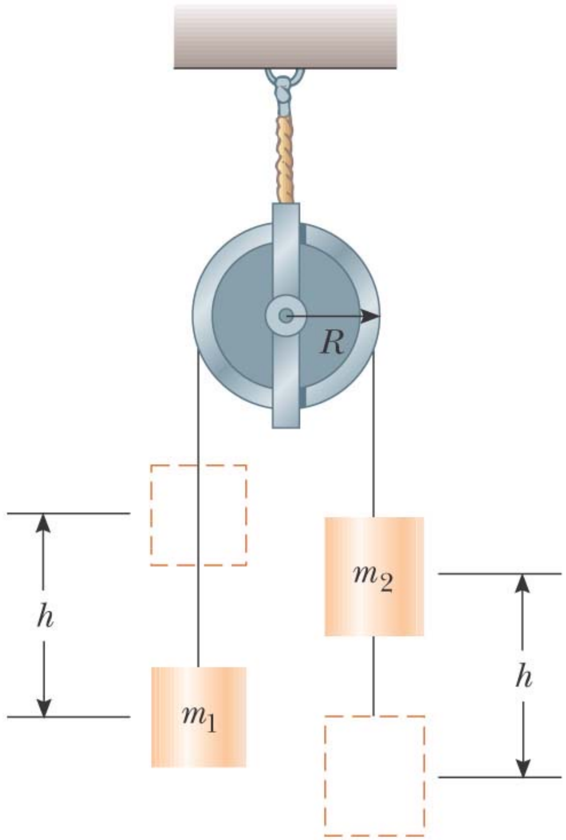
$$\sum W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

# Work-Kinetic Energy Theorem, General

- The rotational form can be combined with the linear form which indicates *the net work done by external forces on an object is the change in its* ***total*** *kinetic energy, which is the sum of the translational and rotational kinetic energies*

# Energy in an Atwood Machine, Example

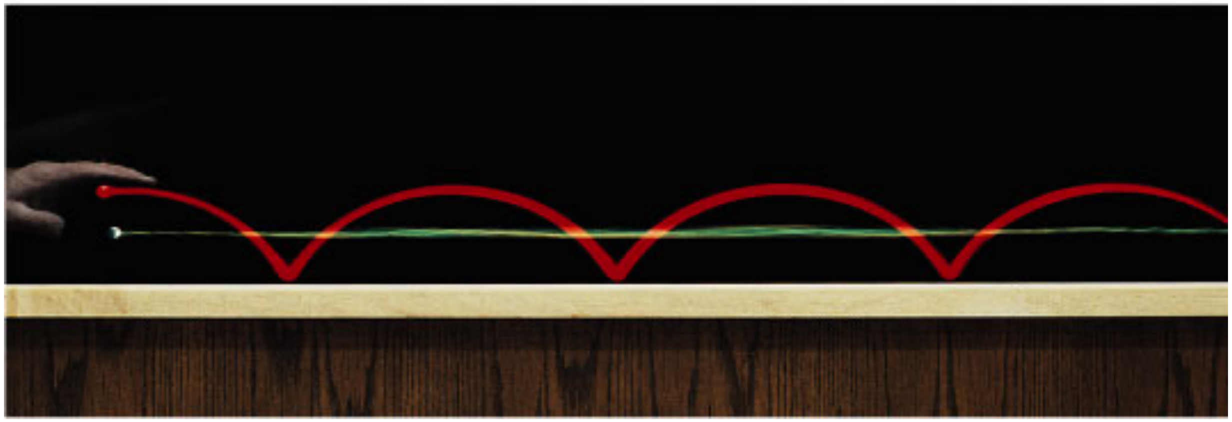
- The blocks undergo changes in translational kinetic energy and gravitational potential energy
- The pulley undergoes a change in rotational kinetic energy



# Summary of Useful Equations

| Rotational Motion About a Fixed Axis | Linear Motion |
|---|---|
| Angular speed $\omega = d\theta/dt$ | Linear speed $v = dx/dt$ |
| Angular acceleration $\alpha = d\omega/dt$ | Linear acceleration $a = dv/dt$ |
| Net torque $\Sigma\tau = I\alpha$ | Net force $\Sigma F = ma$ |
| If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$ | If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$ |
| Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$ | Work $W = \int_{x_i}^{x_f} F_x\, dx$ |
| Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$ | Kinetic energy $K = \frac{1}{2}mv^2$ |
| Power $\mathcal{P} = \tau\omega$ | Power $\mathcal{P} = Fv$ |
| Angular momentum $L = I\omega$ | Linear momentum $p = mv$ |
| Net torque $\Sigma\tau = dL/dt$ | Net force $\Sigma F = dp/dt$ |

# Rolling Object



- The red curve shows the path moved by a point on the rim of the object
  - This path is called a ***cycloid***
- The green line shows the path of the center of mass of the object

# Pure Rolling Motion

- In pure rolling motion, an object rolls without slipping
- In such a case, there is a simple relationship between its rotational and translational motions

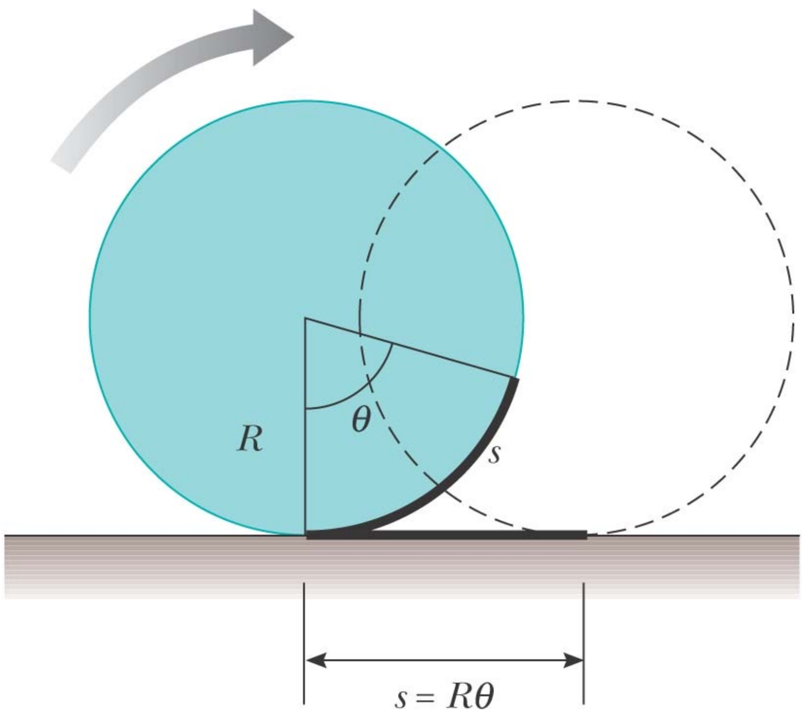
# Rolling Object, Center of Mass

- The velocity of the center of mass is

$$v_{\text{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

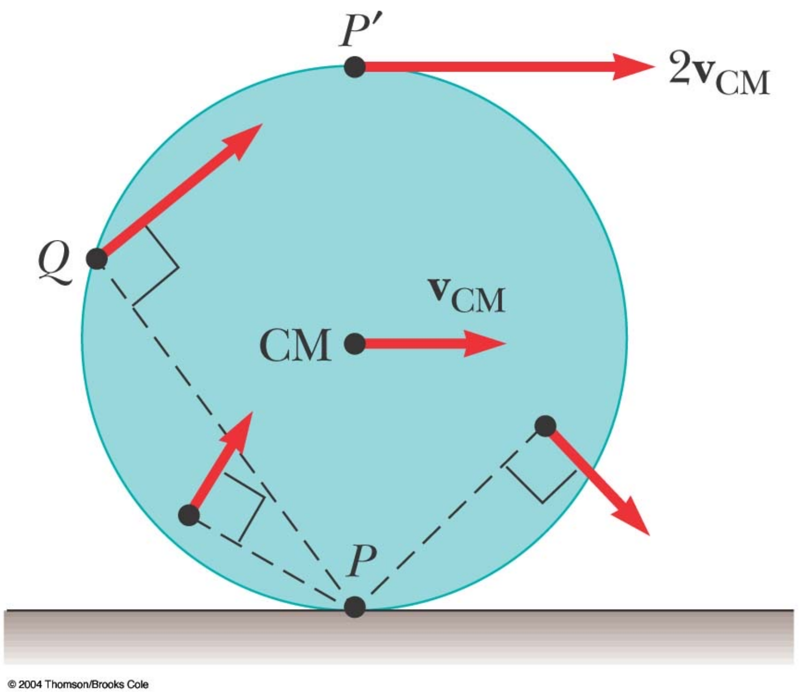
- The acceleration of the center of mass is

$$a_{\text{CM}} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



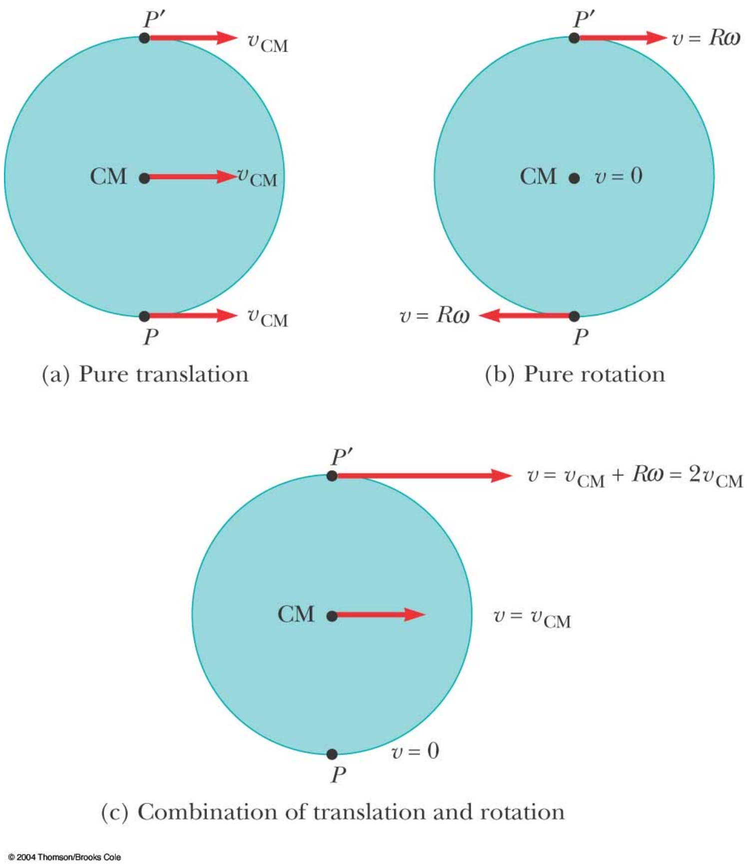
# Rolling Object, Other Points

- A point on the rim, *P*, rotates to various positions such as *Q* and *P*'
- At any instant, the point on the rim located at point *P* is at rest relative to the surface since no slipping occurs



# Rolling Motion Cont.

- Rolling motion can be modeled as a combination of pure translational motion and pure rotational motion

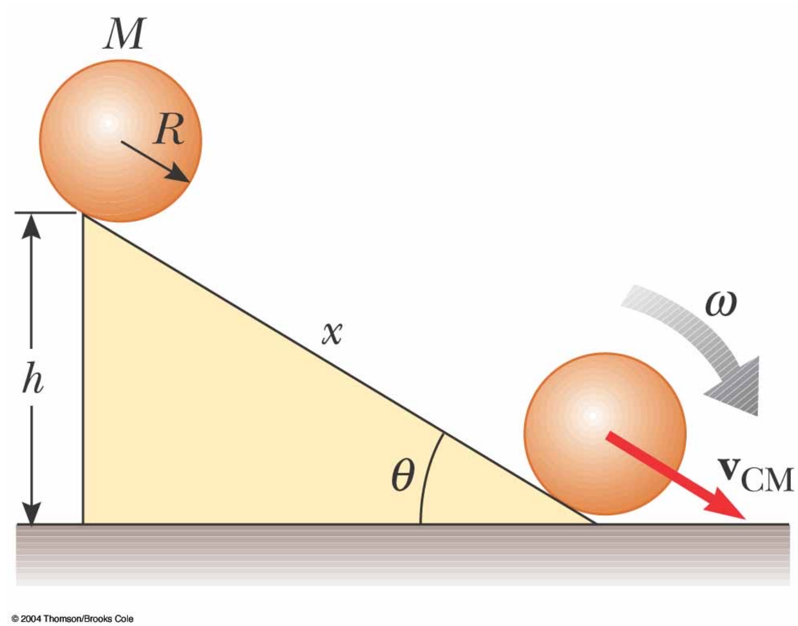


# Total Kinetic Energy of a Rolling Object

- The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass
  - $K = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} M v_{CM}^2$

# Total Kinetic Energy, Example

- Accelerated rolling motion is possible only if friction is present between the sphere and the incline
  - The friction produces the net torque required for rotation



# Total Kinetic Energy, Example cont

- Despite the friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant
- Let $U = 0$ at the bottom of the plane
- $K_f + U_f = K_i + U_i$
- $K_f = ½ (I_{CM} / R^2) v_{CM}^2 + ½ M v_{CM}^2$
- $U_i = Mgh$
- $U_f = K_i = 0$