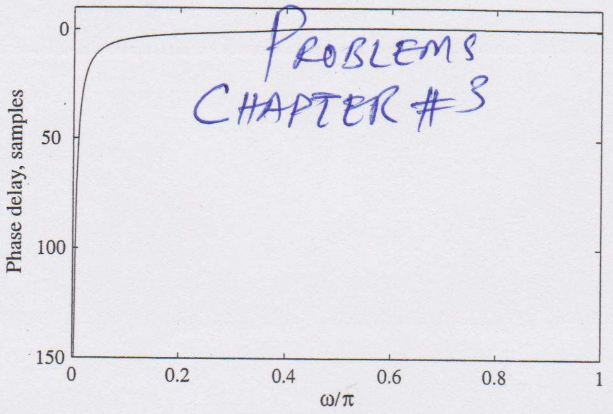

Figure 3.17: Phase delay of the frequency response function of Example 3.21.

Likewise, the group delay can be determined in MATLAB using the M-file `grpdelay`. Several options are also available with this function. We illustrate its use in Example 3.22.

EXAMPLE 3.22 Group Delay Computation Using MATLAB

Using MATLAB, we evaluate the group delay of the frequency response function of Example 3.21. The MATLAB code fragments used to compute the phase delay vector are

```
num = 0.136728736*[1 0 -1];
den = [1 -0.53353098 0.726542528];
[gd,w] = grpdelay(num,den,1024);
```

The plot of the group delay evaluated using MATLAB is shown in Figure 3.18.

3.10 Summary

This chapter provided a short review of the continuous-time Fourier transform (CTFT) representations of continuous-time signals and systems. The discrete-time Fourier transform (DTFT) and its inverse are introduced next along with a discussion of the convergence of the DTFT. Properties of the DTFT are reviewed and the unwrapping of the phase function to remove certain discontinuities in the DTFT is discussed. The concept of the frequency response of a linear, time-invariant (LTI) discrete-time system is then introduced followed by a careful examination of the difference between phase and group delays associated with the frequency response.

3.11 Problems

3.1 Show that the absolute value of the CTFT $X_a(j\Omega)$ defined in Eq. (3.1) is finite if $x_a(t)$ is absolutely integrable.

3.2 Determine the CTFT of the following continuous-time functions defined for $-\infty < t < \infty$:
(a) $y_a(t) = \cos(\Omega_o t)$, (b) $u_a(t) = e^{-\alpha|t|}$, (c) $v_a(t) = e^{j\Omega_o t}$, (d) $p_a(t) = \sum_{\ell=\infty}^{\infty} \delta(t - \ell T)$.

3.3 Determine the CTFT of the following continuous-time functions defined for $-\infty < t < \infty$:

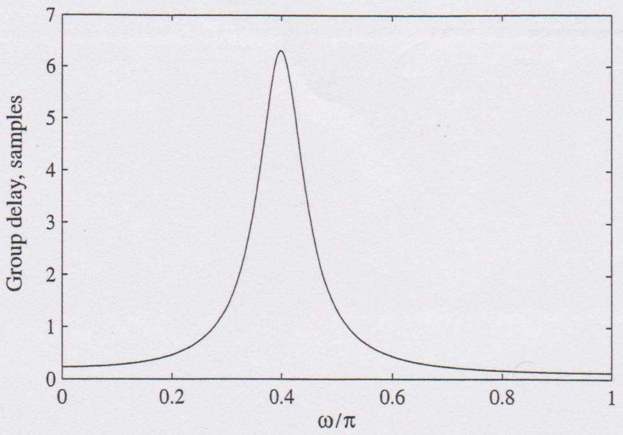

Figure 3.18: Group delay of the frequency response function of Example 3.21.

(a) $v_a(t) = 1$, (b) $\mu(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$ (c) $x_a(t) = \begin{cases} 1, & |t| < \frac{1}{2}, \\ \frac{1}{2}, & |t| = \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}, \end{cases}$ (d) $y_a(t) = \begin{cases} 1 - 2|t|, & |t| < \frac{1}{2}, \\ 0, & |t| \geq \frac{1}{2}, \end{cases}$

3.4 The Gaussian density function, defined in Eq. (A.6), repeated below for convenience

$$h(t) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(t-\mu)^2/2\sigma^2}, \tag{3.123}$$

where σ and μ are, respectively, the variance and the mean of the density function. A continuous-time filter with an impulse response as given by Eq. (3.123) with zero mean is called a *Gaussian filter.* Show that the CTFT of $h(t)$ is also a Gaussian function of Ω.

3.5 The finite-energy function $x_a(t) = \sin(t)/\pi t$ is not absolutely summable. Show that its CTFT is given by

$$X_a(j\Omega) = \begin{cases} 1, & |\Omega| \leq 1, \\ 0, & |\Omega| > 1. \end{cases}$$

3.6 Consider the CTFT pair

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega).$$

Prove the following properties of the CTFT.

(a) **Time-shifting Property:** $x_a(t - t_o) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)e^{-j\Omega t_o}$,

(b) **Frequency-shifting Property:** $x_a(t)e^{j\Omega_o t} \overset{\text{CTFT}}{\longleftrightarrow} X_a(j(\Omega - \Omega_o))$,

(c) **Symmetry Property:** $X_a(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x_a(-j\Omega)$,

(d) **The Scaling Property:** $x_a(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X_a\left(j\frac{\Omega}{a}\right)$,

(e) **Time Differentiation Property:** $\frac{dx_a(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\Omega X_a(j\Omega)$.

3.7 Let $X_a(j\Omega)$ denote the CTFT of a real-valued continuous-time function $x_a(t)$. Show that the magnitude spectrum $|X_a(j\Omega)|$ is an even function of Ω and the phase spectrum $\theta(\Omega) = \arg\{X_a(j\Omega)\}$ is an odd function of Ω.

3.8 Show that the CTFT of the Hilbert transformer defined by Eq. (1.4) is

$$H_{\text{HT}}(j\Omega) = \begin{cases} -j, & \Omega > 0, \\ j, & \Omega < 0. \end{cases}$$

3.9 Let $x(t)$ be a real-valued input signal with a CTFT $X(j\Omega) = X_p(j\Omega) + X_n(j\Omega)$, where $X_p(j\Omega)$ is the portion of $X(j\Omega)$ occupying the positive frequency range and $X_n(j\Omega)$ is the portion of $X(j\Omega)$ occupying the negative frequency range. Let $\hat{x}(t)$ denote the output of an Hilbert transformer with an input $x(t)$. Show that the CTFT $Y(j\Omega)$ of the complex-valued signal $y(t) = x(t) + j\hat{x}(t)$ is given by $Y(j\Omega) = 2X_p(j\Omega)$. Thus, the spectrum of $y(t)$ occupies only the positive frequency range.

3.10 Compute the total energy of the continuous-time signal of Eq. (3.4) with $\alpha = 0.5$ and determine its 80% bandwidth.

3.11 Show that the DTFT of $\mu[n]$ is given by $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$.

3.12 Show that the DTFT of the sequence $x[n] = 1, \ -\infty < n < \infty$, is given by $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$.

3.13 Determine the DTFT of the two-sided sequence $y[n] = \alpha^{|n|}, \ |\alpha| < 1$.

3.14 In Example 3.8, we showed that the inverse DTFT $h_{LP}[n]$ of the DTFT $H_{LP}(e^{j\omega})$ shown in Figure 3.5 is given by Eq. (3.49). Determine and plot the DTFT of $g[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}, \ -\infty < n < \infty$.

3.15 Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$.

(a) Show that if $x[n]$ is even, then it can be computed from $X(e^{j\omega})$ using $x[n] = \frac{1}{\pi}\int_0^{\pi} X(e^{j\omega}) \cos \omega n \, d\omega$.

(b) Show that if $x[n]$ is odd, then it can be computed from $X(e^{j\omega})$ using $x[n] = \frac{j}{\pi}\int_0^{\pi} X(e^{j\omega}) \sin \omega n \, d\omega$.

3.16 Determine the DTFT of the causal sequence $x[n] = A\alpha^n \sin(\omega_0 n + \phi)\mu[n]$, where A, α, ω_0, and ϕ are real, and $|\alpha| < 1$.

3.17 Determine the DTFT of each of the following sequences:
(a) $x_1[n] = \alpha^n \mu[n-1], \quad |\alpha| < 1,$ (b) $x_2[n] = n\alpha^n \mu[n], \quad |\alpha| < 1,$ (c) $x_3[n] = \alpha^n \mu[n+1], \quad |\alpha| < 1,$
(d) $x_4[n] = n\alpha^n \mu[n+2], \quad |\alpha| < 1,$ (e) $x_5[n] = \alpha^n \mu[-n-1], \quad |\alpha| > 1,$ (f) $x_6[n] = \begin{cases} \alpha^{|n|}, & |n| \leq M, \\ 0, & \text{otherwise.} \end{cases}$

3.18 Determine the DTFT of each of the following sequences:
(a) $x_a[n] = \mu[n] - \mu[n-5],$ (b) $x_b[n] = \alpha^n \left(\mu[n] - \mu[n-8]\right), |\alpha| < 1,$ (c) $x_c[n] = (n+1)\alpha^n \mu[n], \ |\alpha| < 1.$

3.19 Determine the DTFT of each of the following finite-length sequences:
(a) $y_1[n] = \begin{cases} 1, & -N \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases}$ (b) $y_2[n] = \begin{cases} 1, & 0 \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases}$ (c) $y_3[n] = \begin{cases} 1 - \frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases}$
(d) $y_4[n] = \begin{cases} N + 1 - |n|, & -N \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases}$ (e) $y_f[n] = \begin{cases} \cos(\pi n/2N), & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$

3.20 Show that the inverse DTFT of

$$X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^m}, \quad |\alpha| < 1,$$

is given by

$$x[n] = \frac{(n + m - 1)!}{n!(m-1)!}\alpha^n \mu[n].$$

3.21 Evaluate the inverse DTFT of each of the following DTFTs:
(a) $X_a(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$, (b) $X_b(e^{j\omega}) = \dfrac{e^{j\omega}(1 - e^{j\omega N})}{1 - e^{j\omega}}$,
(c) $X_c(e^{j\omega}) = 1 + 2\sum_{\ell=0}^{N} \cos\omega\ell$, (d) $X_d(e^{j\omega}) = \dfrac{-\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$, $|\alpha| < 1$.

3.22 Determine the inverse DTFT of each of the following DTFTs:
(a) $H_a(e^{j\omega}) = \sin(4\omega)$, (b) $H_b(e^{j\omega}) = \cos(4\omega)$, (c) $H_c(e^{j\omega}) = \sin(5\omega)$, (d) $H_d(e^{j\omega}) = \cos(5\omega)$.

3.23 Determine the inverse DTFT of each of the following DTFTs:
(a) $H_1(e^{j\omega}) = 1 + 2\cos\omega + 3\cos 2\omega$, (b) $H_2(e^{j\omega}) = (3 + 2\cos\omega + 4\cos 2\omega)\cos(\omega/2)e^{-j\omega/2}$,
(c) $H_3(e^{j\omega}) = j(3 + 4\cos\omega + 2\cos 2\omega)\sin\omega$, (d) $H_4(e^{j\omega}) = j(4 + 2\cos\omega + 3\cos 2\omega)\sin(\omega/2)e^{j\omega/2}$.

3.24 Prove the following theorems of the discrete-time Fourier transform: (a) Linearity theorem, (b) Time-reversal theorem, (c) Time-shifting theorem, and (d) Frequency-shifting theorem.

3.25 Determine and plot the DTFT of the cascade of the LTI discrete-time systems with two-sided impulse responses given by $h_1[n] = \delta[n] - \frac{\sin\omega_1 n}{\pi n}$, and $h_2[n] = \frac{\sin\omega_2 n}{\pi n}$, respectively, where $0 < \omega_1 < \omega_2 < \pi$.

3.26 Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$. Express the inverse DTFT $y[n]$ of $Y(e^{j\omega}) = X(e^{j4\omega})$ in terms of $x[n]$.

3.27 Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$. Define $Y(e^{j\omega}) = \frac{1}{2}\left\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\right\}$. Determine the inverse DTFT $y[n]$ of $Y(e^{j\omega})$.

3.28 Prove Eq. (3.26).

3.29 Prove Eq. (3.33).

3.30 The magnitude function $|X(e^{j\omega})|$ of a discrete-time sequence $x[n]$ is shown in Figure P3.1 for a portion of the angular frequency axis. Sketch the magnitude function for the frequency range $-\pi \le \omega < \pi$. What type of sequence is $x[n]$?

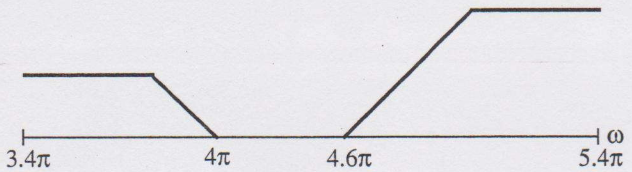

Figure P3.1

3.31 Without computing the DTFT, determine which of the following sequences have real-valued DTFTs and which have imaginary-valued DTFTs:

(a) $x_1[n] = \begin{cases} n, & -N \le n \le N, \\ 0, & \text{otherwise}, \end{cases}$ (b) $x_2[n] = \begin{cases} n^2, & -N \le n \le N, \\ 0, & \text{otherwise}, \end{cases}$ (c) $x_3[n] = \frac{\sin\omega_c n}{\pi n}$,

(d) $x_4[n] = \begin{cases} 0, & \text{for } n \text{ even}, \\ \frac{2}{\pi n}, & \text{for } n \text{ odd}, \end{cases}$ (e) $x_5[n] = \begin{cases} 0, & n = 0, \\ \frac{\cos\pi n}{n}, & |n| > 0. \end{cases}$

3.32 Without computing the inverse DTFT, determine which of the following DTFTs have an inverse that is an even sequence and which have an inverse that is an odd sequence:

(a) $Y_1(e^{j\omega}) = \begin{cases} |\omega|, & 0 \le |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases}$ (b) $Y_2(e^{j\omega}) = j\omega, \quad 0 \le |\omega| \le \pi,$ (c) $Y_3(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0 \\ -j & 0 < \omega < \pi. \end{cases}$

3.33 Without computing the inverse DTFT, determine which of the DTFTs of Figure P3.2 has an inverse that is an even sequence and which has an inverse that is an odd sequence.

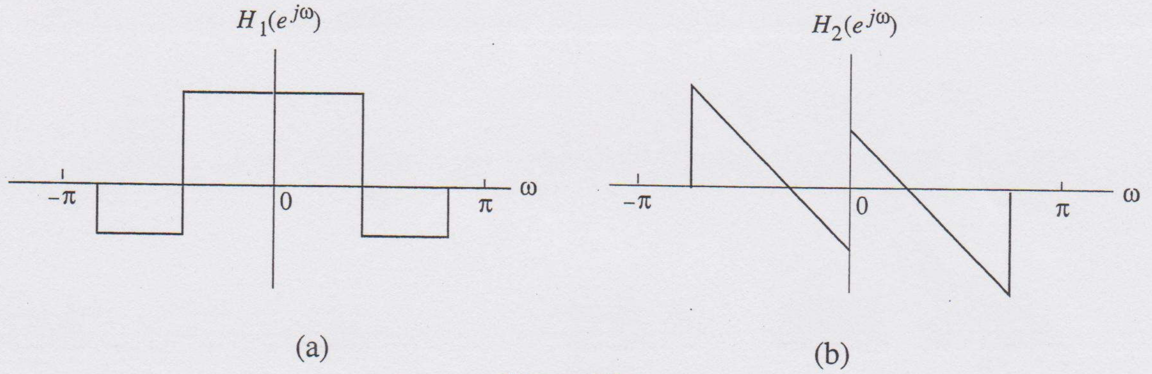

Figure P3.2

3.34 Let $X(e^{j\omega})$ denote the DTFT of a complex sequence $x[n]$. Determine the DTFT $Y(e^{j\omega})$ of the sequence $y[n] = x[n] \circledast x^*[-n]$ in terms of $X(e^{j\omega})$, and show that it is a real-valued function of ω.

3.35 A sequence $x[n]$ has a zero-phase DTFT $X(e^{j\omega})$ as sketched in Figure P3.3. Sketch the DTFT of the sequence $x[n]e^{-j\pi n/3}$.

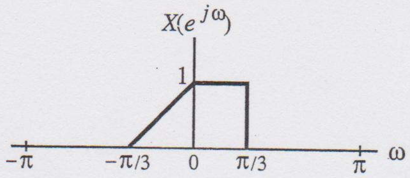

Figure P3.3

3.36 Using Parseval's relation, evaluate the following integrals: (a) $\int_0^\pi \frac{4}{5+4\cos\omega} d\omega$, (b) $\int_0^\pi \frac{1}{3.25 - 3\cos\omega} d\omega$, and (c) $\int_0^\pi \frac{4}{(5-4\cos\omega)^2} d\omega$.

3.37 Let $X(e^{j\omega})$ denote the DTFT of the sequence:

$$x[n] = \{1.2 \quad 2.9 \quad -4.2 \quad 2.4 \quad -3.2 \quad -0.9 \quad 4.4 \quad 4.2 \quad -0.8 \quad 3.9\}, -3 \le n \le 6.$$

Evaluate the integral

$$\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega$$

without computing $X(e^{j\omega})$.

3.38 Let $X(e^{j\omega})$ denote the DTFT of a length-9 sequence $x[n]$ given by

$$x[n] = \{2 \quad 3 \quad -1 \quad 0 \quad -4 \quad 3 \quad 1 \quad 2 \quad 4\}, \; -2 \le n \le 6.$$

Evaluate the following functions of $X(e^{j\omega})$ without computing the transform itself:
(a) $X(e^{j0})$, (b) $X(e^{j\pi})$, (c) $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$, (d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$, (e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$.

3.39 Repeat Problem 3.38 for the length-9 sequence

$$x[n] = \{4 \quad 2 \quad -1 \quad 5 \quad -3 \quad 1 \quad -2 \quad 4 \quad 2\}, \; -6 \le n \le 2.$$

3.40 (a) A measure of the *time delay* of a sequence $x[n]$ is usually given by its *center of gravity*, defined by

$$C_g = \frac{\sum_{n=-\infty}^{\infty} n x[n]}{\sum_{n=-\infty}^{\infty} x[n]}.$$

Express C_g in terms of the DTFT $X(e^{j\omega})$ of $x[n]$.
(b) Determine the center of gravity of the sequence $x[n] = \alpha^n \mu[n]$.

3.41 Let $G_1(e^{j\omega})$ denote the discrete-time Fourier transform of the sequence $g_1[n]$ shown in Figure P3.4(a). Express the DTFTs of the remaining sequences in Figure P3.4 in terms of $G_1(e^{j\omega})$. Do not evaluate $G_1(e^{j\omega})$.

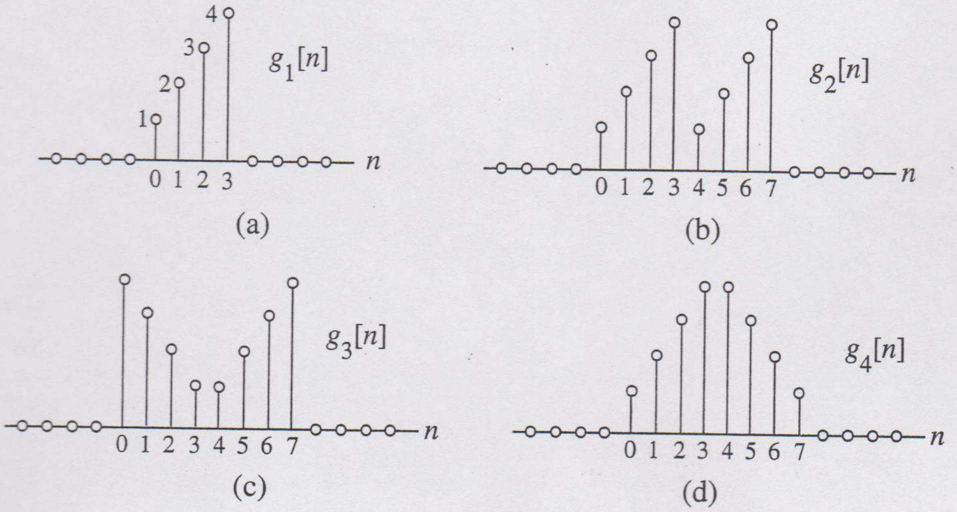

Figure P3.4

3.42 Let $y[n]$ denote the linear convolution of two sequences, $h[n]$ and $x[n]$; that is, $y[n] = h[n] \circledast x[n]$. Show that

(a) $\sum_{n=-\infty}^{\infty} y[n] = \left(\sum_{n=-\infty}^{\infty} h[n]\right)\left(\sum_{n=-\infty}^{\infty} x[n]\right)$, (b) $\sum_{n=-\infty}^{\infty} x[n] = \left(\sum_{n=-\infty}^{\infty} y[n]\right) \Big/ \left(\sum_{n=-\infty}^{\infty} h[n]\right)$,

(c) $\sum_{n=-\infty}^{\infty} (-1)^n y[n] = \left(\sum_{n=-\infty}^{\infty} (-1)^n h[n]\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n x[n]\right)$.

3.43 Show that the 80% bandwidth of the discrete-time signal of Eq. (3.14) for $\alpha = 0.5$ is 0.5081π radians.

3.44 Let $x[n]$ be a causal and absolutely summable real sequence with a DTFT $X(e^{j\omega})$. If $X_{\text{re}}(e^{j\omega})$ and $X_{\text{im}}(e^{j\omega})$ denote the real and imaginary parts of $X(e^{j\omega})$, show that they are related as

$$X_{\text{im}}(e^{j\omega}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} X_{\text{re}}(e^{j\nu})\cot\left(\frac{\omega-\nu}{2}\right)d\nu, \tag{3.124a}$$

$$X_{\text{re}}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{\text{im}}(e^{j\nu})\cot\left(\frac{\omega-\nu}{2}\right)d\nu + x[0]. \tag{3.124b}$$

The above equations are called the *discrete Hilbert transform relations.*

3.45 Show that the function $u[n] = z^n$, where z is a complex constant, is an eigenfunction of an LTI discrete-time system. Is $v[n] = z^n\mu[n]$ with z a complex constant also an eigenfunction of an LTI discrete-time system?

3.46 Determine the expression for the frequency response of the LTI discrete-time system of Figure 2.35 in terms of the frequency responses $H_i(e^{j\omega})$, $1 \le i \le 4$, of the individual blocks.

3.47 Determine the expression for the frequency response of each of the LTI discrete-time systems of Figure P2.2 (see Problem 2.64), in terms of the frequency responses $H_i(e^{j\omega})$, $1 \le i \le 5$, of the individual blocks.

3.48 Determine the expression for the frequency response of the LTI discrete-time system of Figure P2.3 (see Problem 2.65).

3.49 (a) Consider an LTI discrete-time system with a real and causal impulse response $h[n]$ and a frequency response $H(e^{j\omega})$. Show that $h[n]$ can be determined uniquely from the real part $H_{\text{re}}(e^{j\omega})$ of $H(e^{j\omega})$.

(b) The real part of the frequency response of a real and causal LTI discrete-time system is given by $H_{\text{re}}(e^{j\omega}) = 1 + 2\cos\omega + 3\cos 2\omega + 4\cos 3\omega$. Determine the impulse response $h[n]$ of the system.

3.50 If the input to each of the following discrete-time systems is $x[n] = \cos(\omega_o n)$, determine the frequencies present in their outputs:

(a) $y_a[n] = \sin(\pi n/3)x[n]$, (b) $y_b[n] = x^3[n]$, (c) $y_c[n] = x[3n]$.

3.51 Determine a closed-form expression for the frequency response $H(e^{j\omega})$ of the LTI discrete-time system characterized by an impulse response

$$h[n] = \delta[n] - \alpha\delta[n-R], \tag{3.125}$$

where $|\alpha| < 1$. What are the maximum and the minimum values of its magnitude response? How many peaks and dips of the magnitude response occur in the range $0 \le \omega < 2\pi$? What are the locations of the peaks and the dips? Sketch the magnitude and the phase responses for $R = 6$.

3.52 Determine a closed-form expression for the frequency response $G(e^{j\omega})$ of an LTI discrete-time system with an impulse response given by

$$g[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\alpha| < 1$. What is the relation of $G(e^{j\omega})$ to $H(e^{j\omega})$ of Eq. (3.98)? Scale the impulse response by multiplying it with a suitable constant so that the dc value of the magnitude response is unity.

3.53 A noncausal LTI FIR discrete-time system is characterized by an impulse response $h[n] = a_1\delta[n-2] + a_2\delta[n-1] + a_3\delta[n] + a_4\delta[n+1]$. For what values of the impulse response samples will its frequency response $H(e^{j\omega})$ have a zero phase?

3.54 A causal LTI FIR discrete-time system is characterized by an impulse response $h[n] = a_1\delta[n] + a_2\delta[n-1] + a_3\delta[n-2] + a_4\delta[n-3] + a_5\delta[n-4]$. For what values of the impulse response samples will its frequency response $H(e^{j\omega})$ have a linear phase?

3.55 An FIR LTI discrete-time system is described by the difference equation

$$y[n] = a_1 x[n+k] + a_2 x[n+k-1] + a_2 x[n+k-2] + a_1 x[n+k-3],$$

where $y[n]$ and $x[n]$ denote, respectively, the output and the input sequences. Determine the expression for its frequency response $H(e^{j\omega})$. For what values of the constant k will the system have a frequency response $H(e^{j\omega})$ that is a real function of ω?

3.56 Consider the cascade of three causal LTI systems: $h_1[n] = a\delta[n] + b\delta[n-1] + \delta[n-2]$; $h_2[n] = c^n \mu[n]$, $|c| < 1$; and $h_3[n] = d^n \mu[n]$, $|d| < 1$. Determine the frequency response $H(e^{j\omega})$ of the overall system. For what values of the constants $a, b, c,$ and d will $|H(e^{j\omega})| = 1$?

3.57 The input–output relation of a nonlinear discrete-time system in the frequency domain is given by

$$Y(e^{j\omega}) = |X(e^{j\omega})|^{\alpha} e^{j \arg X(e^{j\omega})}, \tag{3.126}$$

where $0 < \alpha \leq 1$, and $X(e^{j\omega})$ and $Y(e^{j\omega})$ denote the DTFTs of the input and output sequences. Determine the expression for its frequency response $H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega})$, and show that it has zero phase. The nonlinear algorithm described by Eq. (3.126) is known as the *alpha-rooting method* and has been used in image enhancement [Jai89].

3.58 Determine the expression for the frequency response $H(e^{j\omega})$ of a causal IIR LTI discrete-time system characterized by the input–output relation

$$y[n] = x[n] - \alpha y[n-R], \qquad |\alpha| < 1,$$

where $y[n]$ and $x[n]$ denote, respectively, the output and the input sequences. Determine the maximum and the minimum values of its magnitude response. How many peaks and dips of the magnitude response occur in the range $0 \leq \omega < 2\pi$? What are the locations of the peaks and the dips? Sketch the magnitude and the phase responses for $R = 5$.

3.59 An IIR LTI discrete-time system with input $x[n]$ and output $y[n]$ is described by the difference equation

$$y[n] + a_1 y[n-1] = b_0 x[n] + b_1 x[n-1],$$

where the constants a_1, b_0, and b_1 are real. Determine the expression for its frequency response. For what values of the constants b_i will the magnitude response be a constant for all values of ω?

3.60 Repeat Problem 3.59 when the constants a_1, b_0, and b_1 are complex numbers.

3.61 Determine the difference equation representation of each of the LTI discrete-time with frequency responses as given below:
(a) $H_a(e^{j\omega}) = \operatorname{cosec}(\omega)$, (b) $H_b(e^{j\omega}) = \sec(\omega)$, (c) $H_c(e^{j\omega}) = \cot(\omega)$, (d) $H_d(e^{j\omega}) = \tan(\omega/2)$.

3.62 Determine the input–output relation of a factor-of-L up-sampler in the frequency domain.

3.63 Determine the inverse DTFT of $G(e^{j\omega}) = 1/(1 - \alpha e^{-jL\omega})$, $|\alpha| < 1$, where L is a positive integer.

3.64 Consider an LTI discrete-time system with an impulse response $h[n] = (0.5)^n \mu[n]$. Determine the frequency response $H(e^{j\omega})$ of the system and evaluate its value at $\omega = \pm\pi/5$. What is the steady-state output $y[n]$ of the system for an input $x[n] = \sin(\pi n/3)\mu[n]$?

3.65 An FIR filter of length 3 is defined by a symmetric impulse response; that is, $h[0] = h[2]$. Let the input to this filter be a sum of two cosine sequences of angular frequencies 0.3 rad/samples and 0.6 rad/samples, respectively. Determine the impulse response coefficients so that the filter passes only the low-frequency component of the input.

3.66 An FIR filter of length 3 is defined by a symmetric impulse response; that is, $h[0] = h[2]$. Let the input to this filter be a sum of two cosine sequences of angular frequencies 0.3 rad/samples and 0.6 rad/samples, respectively. Determine the impulse response coefficients so that the filter passes only the high-frequency component of the input.

3.67 Determine the output response $y[n]$ of an LTI discrete-time system with an impulse response

$$h[n] = \frac{\sin\left((n-2)\frac{\pi}{3}\right)}{(n-2)\pi},$$

for an input

$$x[n] = 3\sin\left(\frac{\pi n}{3}\right) + 5\cos\left(\frac{2\pi n}{5}\right).$$

3.68 An FIR filter of length 5 is defined by a symmetric impulse response; that is, $h[n] = h[4-n], 0 \le n \le 4$. Let the input to this filter be a sum of three cosine sequences of angular frequencies: 0.2 rad/samples, 0.5 rad/samples, and 0.8 rad/samples, respectively. Determine the impulse response coefficients so that the filter passes only the midfrequency component of the input.

3.69 The frequency response $H(e^{j\omega})$ of a length-4 FIR filter with real impulse response has the following specific values: $H(e^{j0}) = 2$, $H(e^{j\pi/2}) = 7 - j3$, and $H(e^{j\pi}) = 0$. Determine its impulse response $h[n]$.

3.70 The frequency response $H(e^{j\omega})$ of a length-4 FIR filter with a real and antisymmetric impulse response has the following specific values: $H(e^{j\pi}) = 8$, and $H(e^{j\pi/2}) = -2 + j2$. Determine its impulse response $h[n]$.

3.71 (a) Design a length-3 FIR notch filter with a symmetric impulse response $h[n]$; that is, $h[n] = h[2-n], 0 \le n \le 2$, and with a notch frequency at 0.4π and a 0 dB dc gain.

(b) Determine the exact expression for the frequency response of the filter designed, and plot its magnitude and phase responses.

3.72 (a) Design a length-4 FIR lowpass filter with a symmetric impulse response $h[n]$; that is, $h[n] = h[3-n], 0 \le n \le 3$, satisfying the following magnitude response values: $|H(e^{j0.2\pi})| = 0.8$ and $|H(e^{j0.5\pi})| = 0.5$.

(b) Determine the exact expression for the frequency response of the filter designed, and plot its magnitude and phase responses.

3.73 (a) Design a length-4 FIR highpass filter with an antisymmetric impulse response $h[n]$; that is, $h[n] = -h[3-n], 0 \le n \le 3$, satisfying the following magnitude response values: $|H(e^{j0.5\pi})| = 0.2$ and $|H(e^{j0.8\pi})| = 0.7$.

(b) Determine the exact expression for the frequency response of the filter designed, and plot its magnitude and phase responses.

3.74 (a) Design a length-5 FIR bandpass filter with an antisymmetric impulse response $h[n]$; that is, $h[n] = -h[4-n], 0 \le n \le 4$, satisfying the following magnitude response values: $|H(e^{j0.4\pi})| = 0.8$ and $|H(e^{j0.8\pi})| = 0.2$.

(b) Determine the exact expression for the frequency response of the filter designed, and plot its magnitude and phase responses.

3.75 Consider the two LTI causal digital filters with impulse responses given by

$$h_A[n] = 0.3\delta[n] - \delta[n-1] + 0.3\delta[n-2],$$
$$h_B[n] = 0.3\delta[n] + \delta[n-1] + 0.3\delta[n-2].$$

(a) Sketch the magnitude responses of the two filters and compare their characteristics.

(b) Let $h_A[n]$ be the impulse response of a causal digital filter with a frequency response $H_A(e^{j\omega})$. Define another digital filter whose impulse response $h_C[n]$ is given by

$$h_C[n] = (-1)^n h_A[n], \qquad \text{for all } n.$$

What is the relation between the frequency response $H_C(e^{j\omega})$ of this new filter and the frequency response $H_A(e^{j\omega})$ of the parent filter?

3.76 As indicated in Example 2.45, the trapezoidal integration formula can be represented as an IIR digital filter represented by a difference equation given by

$$y[n] = y[n-1] + \tfrac{1}{2}\{x[n] + x[n-1]\},$$

with $y[-1] = 0$. Determine the frequency response of the above filter.

3.77 A recursive difference equation representation of the Simpson's numerical integration formula is given by [Ham89]

$$y[n] = y[n-2] + \tfrac{1}{3}\{x[n] + 4x[n-1] + x[n-2]\}.$$

Evaluate the frequency response of the above filter and compare it with that of the trapezoidal method of Problem 6.52.

3.78 The frequency response of an LTI FIR discrete-time system is given by $G(e^{j\omega}) = g_0 + g_1 e^{-j\omega} + g_2 e^{-j2\omega} + g_3 e^{-j3\omega}$. For what relations between the coefficients g_0, g_1, g_2, and g_3 will $G(e^{j\omega})$ have a constant group delay?

3.79 Determine the expressions for the group delay of each of the LTI systems whose frequency responses are given below.

(a) $H_a(e^{j\omega}) = a + be^{-j\omega}$, (b) $H_b(e^{j\omega}) = \frac{1}{1+ce^{-j\omega}}$, (c) $H_c(e^{j\omega}) = \frac{a+be^{-j\omega}}{1+ce^{-j\omega}}$, $|c| < 1$,
(d) $H_b(e^{j\omega}) = \frac{1}{(1+ce^{-j\omega})(1+de^{-j\omega})}$, $|c| < 1$, $|d| < 1$.

3.80 Show that the group delay $\tau_g(\omega)$ of an LTI discrete-time system characterized by a frequency response $H(e^{j\omega})$ can be expressed as

$$\tau_g(\omega) = \operatorname{Re}\left\{\frac{j\frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})}\right\}. \tag{3.127}$$

3.81 Let $H(e^{j\omega})$ denote the frequency response of an LTI discrete-time system with an impulse response $h[n]$ and let $G(e^{j\omega})$ denote the Fourier transform of the sequence $nh[n]$. Show that the group delay of the LTI system can be computed using

$$\tau_g(\omega) = \frac{H_{\text{re}}(e^{j\omega})G_{\text{re}}(e^{j\omega}) + H_{\text{im}}(e^{j\omega})G_{\text{im}}(e^{j\omega})}{|H(e^{j\omega})|^2}, \tag{3.128}$$

where $H_{\text{re}}(e^{j\omega})$ and $H_{\text{im}}(e^{j\omega})$ denote the real and imaginary parts of $H(e^{j\omega})$, respectively, and $G_{\text{re}}(e^{j\omega})$ and $G_{\text{im}}(e^{j\omega})$ denote the real and imaginary parts of $G(e^{j\omega})$, respectively.

3.82 Using Eq. (3.128) determine the group delays of the LTI discrete-time systems with frequency responses as given below:

(a) $H_a(e^{j\omega}) = 1 + 0.4\,e^{-j\omega}$, (b) $H_b(e^{j\omega}) = \frac{1}{1+0.6\,e^{-j\omega}}$, (c) $H_c(e^{j\omega}) = \frac{1-0.5\,e^{-j\omega}}{1+0.3\,e^{-j\omega}}$,
(d) $H_d(e^{j\omega}) = \frac{1}{(1-0.3\,e^{-j\omega})(1+0.5\,e^{-j\omega})}$.

3.83 Consider an LTI discrete-time system with an impulse response $h[n] = (-0.5)^n \mu[n]$. Determine the frequency response $H(e^{j\omega})$ of the system, and evaluate its value at $\omega = \pm\pi/5$. What is the steady-state output $y[n]$ of the system for an input $x[n] = \sin(\pi n/5)\mu[n]$?

3.84 An FIR filter of length 3 is defined by a symmetric impulse response, that is, $h[0] = h[2]$. Let the input to this filter be a sum of two cosine sequences of angular frequencies 0.3 rad/samples and 0.7 rad/samples, respectively. Determine the impulse response coefficients so that the filter passes only the low-frequency component of the input.

3.12 MATLAB Exercises

M 3.1 Using Program 3_1, determine and plot the real and imaginary parts and the magnitude and phase spectra of the following DTFT for various values of r and θ:

$$G(e^{j\omega}) = \frac{1}{1 - 2r(\cos\theta)e^{-j\omega} + r^2 e^{-j2\omega}}, \qquad 0 < r < 1.$$

M 3.2 Using Program 3_1, determine and plot the real and imaginary parts and the magnitude and phase spectra of the DTFTs of the sequences of Problem 3.19 for $N = 10$.

M 3.3 Using Program 3_1, determine and plot the real and imaginary parts, and the magnitude and phase spectra of the following DTFTs:

$$\text{(a) } X(e^{j\omega}) = \frac{0.2418(1 + 0.139e^{-j\omega} - 0.3519e^{-j2\omega} + 0.139e^{-j3\omega} + e^{-j4\omega})}{1 + 0.2386e^{-j\omega} + 0.8258e^{-j2\omega} + 0.1393e^{-j3\omega} + 0.4153e^{-j4\omega}},$$

$$\text{(b) } X(e^{j\omega}) = \frac{0.1397(1 - 0.0911e^{-j\omega} + 0.0911e^{-j2\omega} - e^{-j3\omega})}{1 + 1.1454e^{-j\omega} + 0.7275e^{-j2\omega} + 0.1205e^{-j3\omega}}.$$

M 3.4 Using MATLAB, verify the symmetry relations of the DTFT of a complex sequence as listed in Table 3.1.

M 3.5 Using MATLAB, verify the symmetry relations of the DTFT of a real sequence as listed in Table 3.2.

M 3.6 Using MATLAB, verify the following general properties of the DTFT as listed in Table 3.4: (a) linearity, (b) time-shifting, (c) frequency-shifting, (d) differentiation-in-frequency, (e) convolution, (f) modulation, and (g) Parseval's relation. Since all data in MATLAB have to be finite-length vectors, the sequences used to verify the properties are thus restricted to be of finite length.

M 3.7 Write a MATLAB program to compute the group delay using the expression of Problem 3.80 at a prescribed set of discrete frequencies.

M 3.8 Write a MATLAB program to simulate the filter designed in Problem 3.65, and verify its filtering operation.

M 3.9 Write a MATLAB program to simulate the filter designed in Problem 3.68, and verify its filtering operation.