PROBLEMS
CHAPTER # 4

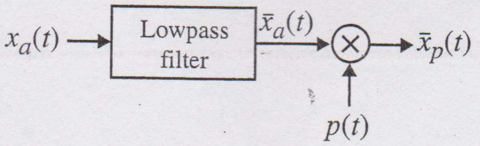


**Figure 4.58:** Equivalent representation of a practical S/H circuit.

Note that the frequency response of the discrete-time system is similar in form to that of the zero-order hold circuit as given in Eq. (4.81) and shown in Figure 4.55(a). Thus, the discrete-time system of Figure 4.58 acts like a narrowband lowpass filter that performs the averaging operation. If the tracking period $\varepsilon$ is much smaller compared to the sampling period $T$, as is usually the case, the effect of the lowpass filter can be neglected, and the practical S/H circuit can be considered as an ideal sampler.

## 4.12 Summary

Various issues concerned with the digital processing of continuous-time signals are studied in this chapter. A discrete-time signal is obtained by uniformly sampling a continuous-time signal. The discrete-time representation is unique if the sampling frequency is greater than twice the highest frequency contained in the continuous-time signal, and the latter can be fully recovered from its discrete-time equivalent by passing it through an ideal analog lowpass reconstruction filter with a cutoff frequency that is half the sampling frequency. If the sampling frequency is lower than twice the highest frequency contained in the continuous-time signal, in general, the latter cannot be recovered from its discrete-time version due to aliasing. In practice, the continuous-time signal is first passed through an analog lowpass anti-aliasing filter, with the cutoff frequency chosen as half of the sampling frequency whose output is sampled to prevent aliasing. It is also shown that a bandpass continuous-time signal can be recovered from its discrete-time equivalent by undersampling, provided the highest frequency is an integer multiple of the bandwidth of the continuous-time signal and the sampling frequency is greater than twice the bandwidth.

A brief review of the theory behind some popular analog lowpass filter design techniques is included, and their design using MATLAB is illustrated. Also discussed are the procedures for designing analog highpass, bandpass, and bandstop filters and their implementations using MATLAB. The specifications of the analog filters are usually given in terms of the locations of the passband and stopband edge frequencies and the passband and stopband ripples. Effects of these parameters on the performances of the anti-aliasing and reconstruction filters are examined.

Other interface devices involved in the digital processing of continuous-time signals are the sample-and-hold circuit, comparator, analog-to-digital converter, and digital-to-analog converter. A brief introduction to these devices is included for completeness.

## 4.13 Problems

**4.1** Prove the Poisson's sum formula of Eq. (4.7).

**4.2** Show that if the spectrum $G_a(j\Omega)$ of $g_a(t)$ (band-limited to $\Omega_m$) also contained an impulse at $\Omega_m$, the sampling rate $\Omega_T$ must be greater than $2\Omega_m$ to recover fully $g_a(t)$ from the sampled version.

**4.3** The Nyquist frequency of a continuous-time signal $g_a(t)$ is $\Omega_m$. Determine the Nyquist frequency of each of the following continuous-time signals derived from $g_a(t)$ :
(a) $y_1(t) = g_a(t)g_a(t)$, (b) $y_2(t) = g_a(t/3)$, (c) $y_3(t) = g_a(3t)$, (d) $y_4(t) = \int_{-\infty}^{\infty} g_a(t-\tau)g_a(\tau)d\tau$,

(e) $y_5(t) = \frac{dg_a(t)}{dt}$.

**4.4** A finite-energy continuous-time signal $g_a(t)$ is sampled at a rate satisfying the Nyquist condition of Eq. (4.11), generating a discrete-time sequence $g[n]$. Develop the relation between the total energy $\mathcal{E}_{g_a(t)}$ of the continuous-time signal $g_a(t)$ and the total energy $\mathcal{E}_{g[n]}$ of the discrete-time signal $g[n]$.

**4.5** A 2.5 s long segment of a continuous-time signal is uniformly sampled without aliasing and generating a finite-length sequence containing 5001 samples. What is the highest frequency component that could be present in the continuous-time signal?

**4.6** A continuous-time signal $x_a(t)$ is composed of a linear combination of sinusoidal signals of frequencies 300 Hz, 500 Hz, 1.2 kHz, 2.15 kHz, and 3.5 kHz. The signal $x_a(t)$ is sampled at a 2.0-kHz rate, and the sampled sequence is passed through an ideal lowpass filter with a cutoff frequency of 900 Hz, generating a continuous-time signal $y_a(t)$. What are the frequency components present in the reconstructed signal $y_a(t)$?

**4.7** A continuous-time signal $x_a(t)$ is composed of a linear combination of sinusoidal signals of frequencies $F_1$ Hz, $F_2$ Hz, $F_3$ Hz, and $F_4$ Hz. The signal $x_a(t)$ is sampled at an 10-kHz rate, and the sampled sequence is then passed through an ideal lowpass filter with a cutoff frequency of 4 kHz, generating a continuous-time signal $y_a(t)$ composed of three sinusoidal signals of frequencies 350 Hz, 575 Hz, and 815 Hz, respectively. What are the possible values of $F_1$, $F_2$, $F_3$, and $F_4$? Is your answer unique? If not, indicate another set of possible values of these frequencies.

**4.8** The continuous-time signal $x_a(t) = 4\sin(20\pi t) - 5\cos(24\pi t) + 3\sin(120\pi t) + 2\cos(176\pi t)$ is sampled at a 50 Hz rate, generating the sequence $x[n]$. Determine the exact expression of $x[n]$.

**4.9** The left and right channels of an analog stereo audio signal are sampled at a 45-kHz rate, with each channel then being converted into a digital bit stream using a 12-bit A/D converter. Determine the combined bit rate of the two channels after sampling and digitization.

**4.10** Show that the impulse response $h_r(t)$ of an ideal lowpass filter as derived in Eq. (4.19) takes the value $h_r(nT) = \delta[n]$ for all $n$ if the cutoff frequency $\Omega_c = \Omega_T/2$, where $\Omega_T$ is the sampling frequency.

**4.11** Consider the system of Figure 4.2, where the input continuous-time signal $x_a(t)$ has a band-limited spectrum $X_a(j\Omega)$, as sketched in Figure P4.1(a), and is being sampled at the Nyquist rate. The discrete-time processor is an ideal lowpass filter with a frequency response $H(e^{j\omega})$, as shown in Figure P4.1(b), and has a cutoff frequency $\omega_c = \Omega_m T/3$, where $T$ is the sampling period. Sketch as accurately as possible the spectrum $Y_a(j\Omega)$ of the output continuous-time signal $y_a(t)$.

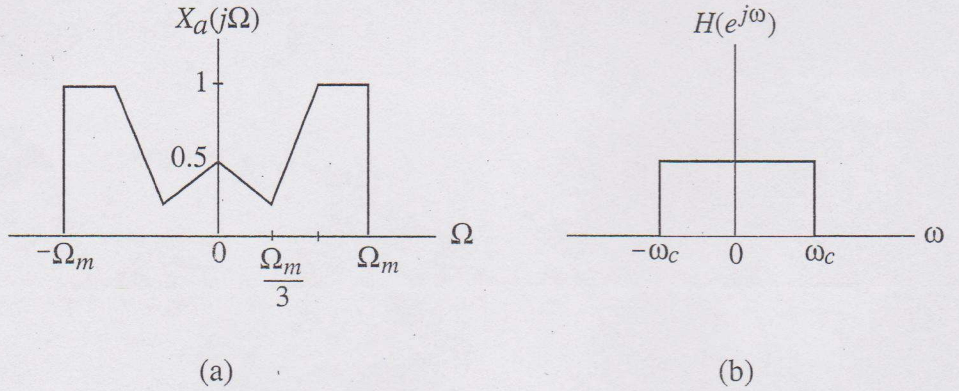


**Figure P4.1**

**4.12** A continuous-time signal $x_a(t)$ has a band-limited spectrum $X_a(j\Omega)$, as indicated in Figure P4.2. Determine the smallest sampling frequency $F_T$ that can be employed to sample $x_a(t)$ so that it can be fully recovered from its sampled version $x[n]$ for each of the following sets of values of the bandedges $\Omega_1$ and $\Omega_2$. Sketch the Fourier transform of the sampled version $x[n]$ obtained by sampling $x_a(t)$ at the smallest sampling rate $F_T$ and the frequency response of the ideal reconstruction filter needed to fully recover $x_a(t)$ for each case.
(a) $\Omega_1 = 100\pi, \Omega_2 = 150\pi$; (b) $\Omega_1 = 160\pi, \Omega_2 = 250\pi$; (c) $\Omega_1 = 110\pi, \Omega_2 = 180\pi$.

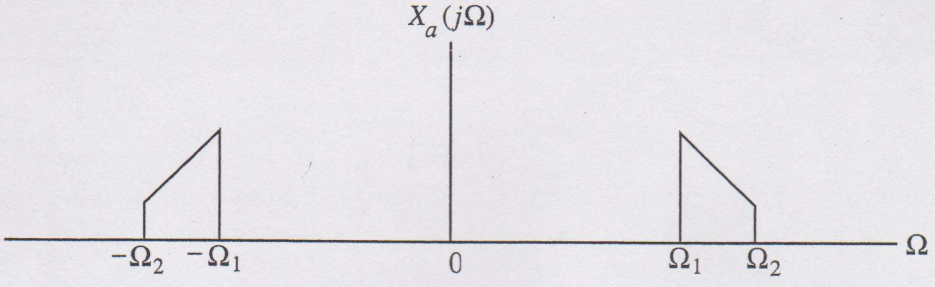


**Figure P4.2**

**4.13** For each set of desired peak passband deviation $\alpha_p$ and the minimum stopband attenuation $\alpha_s$ of an analog lowpass filter given below, determine the corresponding passband and stopband ripples, $\delta_p$ and $\delta_s$ :
(a) $\alpha_p = 0.21$ dB, $\alpha_s = 52$ dB; (b) $\alpha_p = 0.03$ dB, $\alpha_s = 69$ dB; (c) $\alpha_p = 0.33$ dB, $\alpha_s = 57$ dB.

**4.14** Show that the analog transfer function

$$H_a(s) = \frac{a}{s+a}, \qquad a > 0, \tag{4.92}$$

has a lowpass magnitude response with a monotonically decreasing magnitude response with $|H_a(j0)| = 1$ and $|H_a(j\infty)| = 0$. Determine the 3-dB cutoff frequency $\Omega_c$ at which the gain response is 3 dB below the maximum value of 0 dB at $\Omega = 0$.

**4.15** Show that the analog transfer function

$$G_a(s) = \frac{s}{s+a}, \qquad a > 0, \tag{4.93}$$

has a highpass magnitude response with a monotonically increasing magnitude response with $|G_a(j0)| = 0$ and $|G_a(j\infty)| = 1$. Determine the 3-dB cutoff frequency $\Omega_c$ at which the gain response is 3 dB below the maximum value of 0 dB at $\Omega = \infty$.

**4.16** The lowpass transfer function $H_a(s)$ of Eq. (4.92) and the highpass transfer function $G_a(s)$ of Eq. (4.93) can be expressed in the form

$$H_a(s) = \tfrac{1}{2}\{A_0(s) - A_1(s)\}, \qquad G_a(s) = \tfrac{1}{2}\{A_0(s) + A_1(s)\},$$

where $A_0(s)$ and $A_1(s)$ are stable analog allpass transfer functions. Determine $A_0(s)$ and $A_1(s)$.

**4.17** Show that the analog transfer function

$$H_a(s) = \frac{bs}{s^2 + bs + \Omega_o^2}, \qquad b > 0, \tag{4.94}$$

has a bandpass magnitude response with $|H_a(j0)| = |H_a(j\infty)| = 0$ and $|H_a(j\Omega_o)| = 1$. Determine the frequencies $\Omega_1$ and $\Omega_2$ at which the gain is 3 dB below the maximum value of 0 dB at $\Omega_o$. Show that $\Omega_1\Omega_2 = \Omega_o^2$. The difference $\Omega_2 - \Omega_1$ is called the 3-dB bandwidth of the bandpass transfer function. Show that $b = \Omega_2 - \Omega_1$.

**4.18** Show that the analog transfer function

$$G_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + bs + \Omega_o^2}, \qquad b > 0, \tag{4.95}$$

has a bandstop magnitude response with $|G_a(j0)| = |G_a(j\infty)| = 1$ and $|G_a(j\Omega_o)| = 0$. Since the magnitude is exactly zero at $\Omega_o$, it is called the notch frequency, and $G_a(s)$ is often called the notch transfer function. Determine the frequencies $\Omega_1$ and $\Omega_2$ at which the gain is 3 dB below the maximum value of 0 dB at $\Omega = 0$ and $\Omega = \infty$. Show that $\Omega_1\Omega_2 = \Omega_o^2$. The difference $\Omega_2 - \Omega_1$ is called the 3-dB notch bandwidth of the bandpass transfer function. Show that $b = \Omega_2 - \Omega_1$.

**4.19** The bandpass transfer function $H_a(s)$ of Eq. (4.94) and the bandstop transfer function $G_a(s)$ of Eq. (4.95) can be expressed in the form

$$H_a(s) = \tfrac{1}{2}\{A_0(s) - A_1(s)\}, \qquad G_a(s) = \tfrac{1}{2}\{A_0(s) + A_1(s)\},$$

where $A_0(s)$ and $A_1(s)$ are stable analog allpass transfer functions. Determine $A_0(s)$ and $A_1(s)$.

**4.20** An analog real-coefficient allpass transfer function $A(s)$ is defined by $|A(j\Omega)|^2 = 1$, where $A(j\Omega)$ is the magnitude function of the transfer function.

(a) Show that an analog real-coefficient causal and stable allpass transfer function $A(s)$ is given by

$$A(s) = \prod_{i=1}^{N} \left( \frac{s + \lambda_i^*}{s - \lambda_i} \right), \tag{4.96}$$

where $\text{Re}\{\lambda_i\} < 0$.

(b) Show that an analog real-coefficient causal and stable allpass transfer function $A(s)$ satisfies the following property:

$$|A(s)| \begin{cases} < 1 & \text{for Re}(s) > 0, \\ = 1 & \text{for Re}(s) = 0, \\ > 1 & \text{for Re}(s) < 0. \end{cases}$$

**4.21** Show that the first $2N - 1$ derivatives of the squared-magnitude response $|H_a(j\Omega)|^2$ of a Butterworth filter of order $N$ as given by Eq. (4.33) are equal to zero at $\Omega = 0$.

**4.22** Using Eq. (4.35), determine the lowest order of a lowpass Butterworth filter with a 0.25-dB cutoff frequency at 1.5 kHz and a minimum attenuation of 25 dB at 6 kHz. Verify your result using `buttord`.

**4.23** Using Eq. (4.37), determine the pole locations and the coefficients of a sixth-order Butterworth polynomial with unity 3-dB cutoff frequency.

**4.24** Show that the Chebyshev polynomial $T_N(\Omega)$ defined in Eq. (4.40) satisfies the recurrence relation given in Eq. (4.41) with $T_0(\Omega) = 1$, and $T_1(\Omega) = \Omega$.

**4.25** Using Eq. (4.43), determine the lowest order of a lowpass Type 1 Chebyshev filter with a 0.25-dB cutoff frequency at 1.5 kHz and a minimum attenuation of 25 dB at 6 kHz. Verify your result using `cheb1ord`.

**4.26** Using Eq. (4.54) determine the lowest order of a lowpass elliptic filter with a 0.25-dB cutoff frequency at 1.5 kHz and a minimum attenuation of 25 dB at 6 kHz. Verify your result using `ellipord`.

**4.27** Determine the Bessel polynomials $B_N(s)$ for the following values of $N$: (a) $N = 4$ and (b) $N = 5$.

**4.28** The transfer function of a third-order analog Butterworth lowpass filter with a passband edge at 0.24 Hz and a passband ripple of 0.5 dB is given by

$$H_{LP}(s) = \frac{10}{s^3 + 4.309s^2 + 9.2835s + 10}.$$

Determine the transfer function $H_{HP}(s)$ of an analog highpass filter with a passband edge at 3 Hz and a passband ripple of 0.5 dB by applying the spectral transformation of Eq. (4.62).

**4.29** The transfer function of a third-order analog Butterworth highpass filter with a passband edge at 0.9 Hz and a passband ripple of 1 dB is given by

$$H_{HP}(s) = \frac{s^3}{s^3 + 9.283s^2 + 40.087s + 100}.$$

Determine the transfer function $H_{LP}(s)$ of an analog lowpass filter with a passband edge at 3 Hz and a passband ripple of 1 dB by applying the spectral transformation of Eq. (4.62).

**4.30** The transfer function of a second-order analog elliptic lowpass filter with a passband edge at 0.25 Hz and a passband ripple of 0.5 dB is given by

$$H_{LP}(s) = \frac{0.01(s^2 + 367.93)}{s^2 + 2.269s + 3.895}.$$

Determine the transfer function $H_{BP}(s)$ of an analog bandpass filter with a center frequency at 3 Hz and a bandwidth of 0.5 Hz by applying the spectral transformation of Eq. (4.64).

**4.31** A Butterworth analog highpass filter is to be designed with the following specifications: $F_p = 6.5$ kHz, $F_s = 1.5$ kHz, $\alpha_p = 0.5$ dB, and $\alpha_s = 40$ dB. What are the bandedges and the order of the corresponding analog lowpass filter? What is the order of the highpass filter? Verify your results using the function `buttord`.

**4.32** An elliptic analog bandpass filter is to be designed with the following specifications: passband edges at 20 kHz and 45 kHz, stopband edges at 15 kHz and 50 kHz, peak passband ripple of 0.25 dB, and minimum stopband attenuation of 50 dB. What are the bandedges and the order of the corresponding analog lowpass filter? What is the order of the bandpass filter? Verify your results using the function `ellipord`.

**4.33** A Type 1 Chebyshev analog bandstop filter is to be designed with the following specifications: passband edges at 10 MHz and 70 MHz, stopband edges at 20 MHz and 45 MHz, peak passband ripple of 0.5 dB, and minimum stopband attenuation of 30 dB. What are the bandedges and the order of the corresponding analog lowpass filter? What is the order of the bandstop filter? Verify your results using the function `cheb1ord`.

**4.34** Verify Table 4.1.

**4.35** Derive Eq. (4.76).

**4.36** Derive Eq. (4.77).

**4.37** An alternative to the zero-order hold circuit of Figure 4.54 used for signal reconstruction at the output of a D/A converter is the *first-order hold circuit*, which approximates $y_a(t)$ according to the following relation:

$$y_f(t) = y_p(nT) + \frac{y_p(nT) - y_p(nT - T)}{T}(t - nT), \quad nT \le t < (n+1)T.$$

As indicated by the above equation, the first-order hold circuit approximates $y_a(t)$ by straight-line segments. The slope of the segment between $t = nT$ and $t = (n+1)T$ is determined from the sample values $y_p(nT)$ and $y_p(nT - T)$. Determine the impulse response $h_f(t)$ and the frequency response $H_f(j\Omega)$ of the first-order hold circuit, and compare its performance with that of the zero-order hold circuit.

**4.38** A more improved signal reconstruction at the output of a D/A converter is provided by a linear interpolation circuit, which approximates $y_a(t)$ by connecting successive sample points of $y_p(t)$ with straight-line segments. The input–output relation of this circuit is given by

$$y_f(t) = y_p(nT - T) + \frac{y_p(nT) - y_p(nT - T)}{T}(t - nT), \quad nT \leq t < (n+1)T.$$

Determine the impulse response $h_f(t)$ and the frequency response $H_f(j\Omega)$ of the linear interpolation circuit, and compare its performance with that of the first-order hold circuit.

## 4.14 MATLAB Exercises

**M 4.1** Determine the transfer function of a lowpass Butterworth analog filter with specifications as given in Problem 4.22, using Program 4_2. Plot the gain response and verify that the filter designed meets the given specifications. Show all steps.

**M 4.2** Determine the transfer function of a lowpass Type 1 Chebyshev analog filter with specifications as given in Problem 4.25, using Program 4_3. Plot the gain response and verify that the filter designed meets the given specifications. Show all steps.

**M 4.3** Modify Program 4_3 to design lowpass Type 2 Chebyshev analog filters. Using this program, determine the transfer function of a lowpass Type 2 Chebyshev analog filter with specifications as given in Problem 4.25. Plot the gain response and verify that the filter designed meets the given specifications. Show all steps.

**M 4.4** Determine the transfer function of a lowpass elliptic analog filter with specifications as given in Problem 4.26, using Program 4_4. Plot the gain response and verify that the filter designed meets the given specifications. Show all steps.

**M 4.5** Design, using MATLAB, a Butterworth analog highpass filter with specifications given in Problem 4.31. Show the transfer functions of the prototype analog lowpass and the highpass filters. Plot their gain responses and verify that both filters meet their respective specifications. Show all steps.

**M 4.6** Design an elliptic analog bandpass filter with specifications given in Problem 4.32. Show the transfer functions of the prototype analog lowpass and the bandpass filters. Plot their gain responses and verify that both filters meet their respective specifications. Show all steps.

**M 4.7** Design a Type 1 analog bandstop filter with specifications given in Problem 4.33. Show the transfer functions of the prototype analog lowpass and the bandstop filters. Plot their gain responses and verify that both filters meet their respective specifications. Show all steps.

**M 4.8** Write a MATLAB program to verify the plots of Figure 4.56.