** Diagonalization

Definitions :

(1) Diagonal matrix is a square matrix where $a_{ij} = 0$ for all $i \neq j$.

(2) Diagonalizable matrix is a square matrix such that there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. Then, we say that P diagonalizes A

(3) Square matrices A, B are called similar if $B = P^{-1}AP$ for some invertible matrix P.

Our goal in this chapter :
For a square matrix A, does there exists an invertible matrix P such that $P^{-1}AP$ is diagonal matrix ?

Properties :

[1] if A and B are similar matrices then A and B have the same eigen values.

[2] Problem of diagonalization is close to find the eigen vectors.

(Ex) Is $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ a diagonalizable matrix?

Solution

(STEP 1) Find the eigen values of A.

Put $\Delta = 0 \Rightarrow \begin{vmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = 0$

$\Rightarrow (\lambda+2)\left[(\lambda-1)^2 - 9\right] = 0$

$\Rightarrow (\lambda+2)^2(\lambda-4) = 0 \Rightarrow \lambda = 4, \ \lambda = -2$

(STEP 2) The eigen vector respect to $\lambda = 4$:

Put $(A - \lambda I)X = 0$

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

then, by solution the system,

$$P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ (the basis of eigen space)

(STEP 3) The eigen vector respect to $\lambda = -2$:

By same ~~and~~ calculation

$$P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ (the basis of the eigen space)

(STEP 4)

Let $P = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{|P|} \operatorname{adj}(P) \Rightarrow P^{-1} = \begin{bmatrix} \ldots\ldots \end{bmatrix}$

(STEP 5)

$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ diagonal matrix.

★ Remark :

A square matrix $A_{n \times n}$ is diaganalizable iff A has n different eigen vectors.

(Ex) Determin whether $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ is diaganalizable or not ?

Salution

Put $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(-\lambda)(-3-\lambda) = 0$

$\Rightarrow \lambda = \boxed{1}$ or $\boxed{0}$ or $\boxed{-3}$

So, we have 3 distinct eigen vectors
Hence, A is diaganalizable.

Homework:
Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ where
$T(x, y, z) = (x - y - z,\ x + 3y + z,\ -3x + y - z)$.
find the standered matrix and determin whether it is diagonalizable or not ?