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Even Higher Order Fractional Initial Boundary Value Problem with Nonlocal Constraints of Purely Integral Type

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Abstract: In this paper, the a priori estimate method, the so-called energy inequalities method based on some functional analysis tools is developed for a Caputo time fractional $2m$th order diffusion wave equation with purely nonlocal conditions of integral type. Existence and uniqueness of the solution are proved. The proofs of the results are based on some a priori estimates and on some density arguments.

Keywords: energy inequality; integral conditions; fractional wave equation; existence and uniqueness; initial boundary value problem

1. Introduction

Classical initial boundary value problems for partial differential equations with integer and noninteger order have been widely studied during the last three decades by using different methods. One of the most important methods used and applied to linear and nonlinear partial differential equations with integer order supplemented with classical conditions is the functional analysis method. However, for equations with Caputo time fractional order and nonlocal conditions, there are only a few results obtained by using the mentioned method. The Caputo fractional derivative is a nonlocal operator since it is an integral which is a nonlocal operator. Caputo time fractional derivative can be used to model systems with memory, since it requires all the past history. Time fractional order partial differential equations play a great role in reducing the errors coming from the neglected parameters while modeling real life phenomena.

One of the most important classes of the above equations are the fractional diffusion-wave equations that have been studied and used in different branches of Science. Problem (1) constitutes a large class of time fractional diffusion wave equations of even order such as second and fourth order time fractional wave equations that have numerous applications in physics and engineering as mentioned below. In our problem, local conditions at 0 and 1 are replaced by other conditions on the moments of order $1, 2, ..., 2m-1$ which are non-local integral conditions. Although mathematical models in two and three-dimensions are of big significance for applications, most of the recent research articles are devoted to the fractional order diffusion wave equations in one-dimensional settings. These equations model, for example, propagation of mechanical waves in viscoelastic media [1–4], a non-Markovian diffusion process with memory [5], and a model governing the propagation of mechanical diffusive waves in viscoelastic media that exhibit a power-law creep [2–4].

For various applications of fractional calculus, the reader could refer to [4,6–13].

In the literature, many researchers used the functional analysis method to investigate the well posedness of initial boundary value problems for partial differential equations with time and space integer order having nonlocal conditions—we cite, for example, the references [14–17]. For the fractional diffusion wave equations case with higher order derivatives and classical boundary conditions, there are only few papers dealing with the existence and uniqueness of solution such as [18–20]. In this paper, an initial boundary value problem with purely nonlocal constraints of integral type for a Caputo time fractional $2m$th order diffusion wave equation is studied by applying the functional analysis method, the so-called energy inequality method based mainly on some a priori estimates and on the density of the range of the operator generated by the studied problem. This work can be considered as a contribution to the development of the functional analysis method used to prove the well posedness of problems with fractional order. The obtained results show the efficiency of this method to study the existence and uniqueness of solution for the time fractional order differential equations with nonlocal conditions.

This paper is organized as follows: in Section 2, we set our fractional initial boundary value problem. In Section 3, we give some preliminaries concerning the used function spaces, some useful tools and write down the given problem in its operator form. In Section 4, we establish an a priori estimate for the solution and deduce some consequences about the uniqueness of the solution and its dependence on the free term and the given data. Section 5 provides proofs of the main result concerning the solvability of the posed problem. We end our problem with conclusions.

2. Problem Setting

In the domain $Q = (0,1) \times (0,T)$ where $0 \leq T < \infty$, we consider the time fractional initial boundary problem of higher order with purely integral conditions

$$\begin{cases} \mathcal{L}v = \partial_t^{\alpha+1} v + (-1)^m \theta(t) \frac{\partial^{2m} v}{\partial x^{2m}} = f(x,t)\,, & x \in (0,1)\ t \in (0,T), \\ l_1 v = v(x,0) = g(x),\ l_2 v = v_t(x,0) = h(x), & x \in (0,1), \\ \int\limits_0^1 x^i v(x,t)dx = 0\ \ ,\ i = \overline{0, 2m-1}, & t \in (0,T), \end{cases} \tag{1}$$

where $\theta(t)$, $f(x,t)$, $g(x)$ and $h(x)$ are given functions that satisfy certain conditions which will be specified later on, and the operator $\partial_t^{\alpha+1}$ denotes the Caputo left fractional derivative of order $1+\alpha$ with $0 < \alpha < 1$ defined by (see [21])

$$\partial_t^{\alpha+1} v(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{v_{\tau\tau}(x,\tau)}{(t-\tau)^{\alpha}} d\tau,\ t > 0, \tag{2}$$

where $\Gamma(1-\alpha)$ is the Gamma function.

The Riemann–Liouville integral of order $0 < \alpha < 1$ is defined by (see [21])

$$D_t^{-\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{v(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{3}$$

Different properties of the Caputo fractional derivative and Riemann fractional-Liouville integral can be found in [21–23] and the references therein.

3. Preliminaries

In this section, we introduce some important lemmas and inequalities needed throughout the sequel, and write the posed problem in its operator form.

Lemma 1 (Poincare type inequality). *For $m \in \mathbb{N}$, we have*

$$\|\mathcal{I}_x^{2m} v\|_{L^2(0,A)}^2 \leq (\frac{A}{2})^{2m} \|v\|_{L^2(0,A)}^2, \tag{4}$$

where

$$\mathcal{I}_x^{2m} v = \int_0^x \int_0^{\xi_1} ... \int_0^{\xi_{2m-1}} v(\eta, t) d\eta d\xi_{2m-1} ... d\xi_1 = \int_0^x \frac{(x-\xi)^{2m-1}}{(2m-1)!} v(\xi, t) d\xi.$$

Lemma 2 ([24]). *For any absolutely continuous function $J(s)$ on the interval $[0, T]$, the following inequality hold*

$$J(s)\, \partial_s^\alpha J(s) \geq \frac{1}{2} \partial_s^\alpha J^2(s), \qquad 0 < \alpha < 1. \tag{5}$$

Lemma 3 ([25]). *Let $\varphi(t)$ be nonnegative and absolutely continuous on $[0, T]$, and, for almost all $t \in [0, T]$, satisfies the inequality*

$$\frac{d\varphi}{dt} \leq C(t)\varphi(t) + B(t), \tag{6}$$

where the functions $C(t)$ and $B(t)$ are summable and nonnegative on $[0, T]$. Then,

$$\begin{aligned} \varphi(t) &\leq e^{\int_0^t C(\tau)d\tau} \left(\varphi(0) + \int_0^t B(\xi).e^{\int_0^\xi C(\tau)d\tau} d\xi \right) \\ &\leq e^{\int_0^t C(\tau)d\tau} \left(\varphi(0) + \int_0^t B(\tau)d\tau \right). \end{aligned} \tag{7}$$

Lemma 3 can be generalized as

Lemma 4 ([24]). *Let a nonnegative absolutely continuous function $\mathcal{Z}(t)$ satisfy the inequality*

$$\partial_t^\alpha \mathcal{Z}(t) \leq c_1 \mathcal{Z}(t) + c_2(t), \quad 0 < \alpha < 1, \tag{8}$$

for almost all $t \in [0, T]$, where c_1 is a positive constant and $c_2(t)$ is an integrable nonnegative function on $[0, T]$. Then,

$$\mathcal{Z}(t) \leq \mathcal{Z}(0) E_\alpha(c_1 t^\alpha) + \Gamma(\alpha) E_{\alpha,\alpha}(c_1 t^\alpha) D_t^{-\alpha} c_2(t), \tag{9}$$

where

$$E_\alpha(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(\alpha n + 1)} \text{ and } E_{\alpha,\alpha^*}(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(\alpha n + \alpha^*)}, \tag{10}$$

are Mittag–Leffler functions.

Lemma 5 ([14]). *Let $Z_i(\tau)$ $(i = 1, 2, 3)$ be nonnegative functions on the interval $[0, T]$, $Z_1(\tau), Z_2(\tau)$ are integrable functions, and $Z_3(\tau)$ is nondecreasing. Then,*

$$\int_0^t Z_1(\tau) d(\tau) + Z_2(t) \leq Z_3(t) + C \int_0^t Z_2(\tau) d(\tau)$$

implies

$$\int_0^t Z_1(\tau) d(\tau) + Z_2(t) \leq e^{Ct} Z_3(t).$$

Young's inequality with ε: For any $\varepsilon > 0$, we have the inequality

$$\lambda\beta \leq \frac{1}{p}\left|\varepsilon\lambda\right|^p + \frac{p-1}{p}\left|\frac{\beta}{\varepsilon}\right|^{\frac{p}{p-1}}, \qquad \lambda,\beta \in \mathbb{R},\ p > 1, \tag{11}$$

where λ and β are nonnegative numbers.

A special case of (11) is the Cauchy inequality with ε:

$$\lambda\beta \leq \frac{\varepsilon}{2}\lambda^2 + \frac{1}{2\varepsilon}\beta^2, \quad \varepsilon > 0, \tag{12}$$

The solution of the problem (1) can be regarded as the solution of operator equation

$$\mathcal{M}v = \mathcal{W} = (f, g, h), \tag{13}$$

where $\mathcal{M} = (\mathcal{L}, l_1, l_2)$, and $\mathcal{M} : \mathcal{B} \longrightarrow \mathcal{Y}$ is an unbounded operator with domain of definition

$$\mathcal{D}(\mathcal{M}) = \begin{cases} v \in L^2(Q),\ \partial_t^{\alpha+1}v,\ \frac{\partial^{2m}v}{\partial x^{2m}} \in L^2(Q), \\ \int\limits_0^1 x^i v dx = 0,\ \ i = \overline{0, 2m-1},\ t \in (0,T), \end{cases} \tag{14}$$

such that v satisfies the initial conditions and where $\mathcal{B}$ is a Banach space of functions v endowed with the finite norm

$$\|v\|_{\mathcal{B}}^2 = \sup_{0\leq t\leq T}\left(D_t^{\alpha-1}\|\mathcal{I}_x^m v_t(x,t)\|_{L^2(0,1)}^2 + \int\limits_0^1 v^2(x,t)dx\right) \tag{15}$$

and $\mathcal{Y}$ is Hilbert space constituting of the elements $\mathcal{W} = (f, g, h)$ equipped with the norm

$$\|\mathcal{W}\|_Y^2 = \|g\|_{L^2(0,1)}^2 + \|h\|_{L^2(0,1)}^2 + \|f\|_{L^2(Q)}^2. \tag{16}$$

Here, $\mathcal{L}$ denotes the time fractional differential operator

$$\mathcal{L} = {}^C\partial_t^{\alpha+1} + (-1)^m\theta(t)\frac{\partial^{2m}}{\partial x^{2m}}.$$

4. A Priori Estimate for the Solution and Uniqueness

To prove the uniqueness of solution of problem (1), we establish an energy inequality for the solution from which we deduce the uniqueness of solution of the posed problem.

Theorem 1. *Assume that the function $\theta(t)$ satisfies the conditions*

$$i)\ c_2 \leq \theta(t) \leq c_1,\ \ ii)\ c_4 \leq \theta'(t) \leq c_3,\ \ \forall t \in [0,T], \tag{17}$$

where c_1, c_2, c_3, and c_4 are positive constants. Then, for any $v \in \mathcal{D}(\mathcal{M})$, there exists a positive constant $\mathcal{K}$ such that the following a priori estimate is satisfied:

$$\begin{aligned} &\sup_{0\leq t\leq T}\left(D_t^{\alpha-1}\|\mathcal{I}_x^m v_t(x,t)\|_{L^2(0,1)}^2 + \int\limits_0^1 v^2(x,t)dx\right) \\ \leq\ & \mathcal{K}\left(\|g\|_{L^2(0,1)}^2 + \|h\|_{L^2(0,1)}^2 + \|f\|_{L^2(Q)}^2\right), \end{aligned} \tag{18}$$

where $\mathcal{K} = \mathcal{K}(\eta, \delta, \rho)$ is given by

$$\mathcal{K} = \rho\left(1 + \frac{T^\alpha}{\Gamma(1+\alpha)}\right), \tag{19}$$

with η, δ and ρ are respectively given by (30), (34) and (37).

Proof. For $v \in D(L)$, we consider the scalar product in $L^2(0,1)$ of the differential equation in problem (1) and the integrodifferential operator $Nv = 2(-1)^m \mathcal{I}_x^{2m} v_t$, we have

$$\begin{aligned} & 2(-1)^m(\partial_t^{\alpha+1} v, \mathcal{I}_x^{2m} v_t)_{L^2(0,1)} + 2(\theta(t)\frac{\partial^{2m} v}{\partial x^{2m}}, \mathcal{I}_x^{2m} v_t)_{L^2(0,1)} \\ = \; & (\mathcal{L}v, Nv)_{L^2(0,1)}. \end{aligned} \tag{20}$$

We separately consider the inner products on the left-hand side of Equation (20) and we integrate by parts and taking into account boundary and initial conditions in Problem (1), we obtain

$$\begin{aligned} 2(-1)^m(\partial_t^{\alpha+1} v, \mathcal{I}_x^{2m} v_t)_{L^2(0,1)} &= 2(-1)^m(\partial_t^{\alpha} v_t, \mathcal{I}_x^{2m} v_t)_{L^2(0,1)} \\ &= 2(\partial_t^{\alpha}(\mathcal{I}_x^{m} v_t), \mathcal{I}_x^{m} v_t)_{L^2(0,1)}, \end{aligned} \tag{21}$$

$$\begin{aligned} 2\left(\theta(t)\frac{\partial^{2m} v}{\partial x^{2m}}, \mathcal{I}_x^{2m} v_t\right)_{L^2(0,1)} &= 2\int_0^1 \theta(t)\frac{\partial^{2m} v}{\partial x^{2m}} \mathcal{I}_x^{2m} v_t dx \\ &= 2(-1)^m(\theta(t)\frac{\partial^{m} v}{\partial x^{m}}, \mathcal{I}_x^{m} v_t)_{L^2(0,1)} \\ &= 2(\theta(t)v, v_t)_{L^2(0,1)}. \end{aligned} \tag{22}$$

Substitution of (21) and (22) into (20) yields

$$\begin{aligned} & 2(\partial_t^{\alpha}(\mathcal{I}_x^{m} v_t), \mathcal{I}_x^{m} v_t)_{L^2(0,1)} + 2(\theta(t)v, v_t)_{L^2(0,1} \\ = \; & (\mathcal{L}v, Nv)_{L^2(0,1)} = 2(-1)^m(f, \mathcal{I}_x^{2m} v_t)_{L^2(0,1)}. \end{aligned} \tag{23}$$

By Lemmas 1 and 2 and inequality (12), identity (23) reduces to

$$\partial_t^{\alpha}\|\mathcal{I}_x^{m} v_t\|^2_{L^2(0,1)} + (\theta(t)v, v_t)_{L^2(0,1} \leq \|f\|^2_{L^2(0,1)} + \frac{1}{2^m}\|\mathcal{I}_x^{m} v_t\|^2_{L^2(0,1)}. \tag{24}$$

Replacing t by τ , integrating with respect to τ from zero to t and using given conditions, we obtain

$$\begin{aligned} & \int_0^t \partial_\tau^{\alpha}\|\mathcal{I}_x^{m} v_\tau\|^2_{L^2(0,1)} d\tau + \int_0^t\int_0^1 \theta(\tau) v v_\tau dx d\tau \\ \leq \; & \int_0^t \|f(x,\tau)\|^2_{L^2(0,1)} d\tau + \frac{1}{2^m}\int_0^t \|\mathcal{I}_x^{m} v_\tau\|^2_{L^2(0,1)} d\tau. \end{aligned} \tag{25}$$

The second term on the left-hand side can be evaluated as

$$\begin{aligned} 2\int_0^t\int_0^1 \theta(\tau) v v_\tau dx d\tau &= \int_0^1 \theta(t) v^2 dx - \theta(0)\int_0^1 g^2(x) dx \\ & -\int_0^t\int_0^1 \theta'^2 v^2 dx d\tau. \end{aligned} \tag{26}$$

Hence, inequality (25) becomes

$$\begin{aligned}
&\int_0^t \partial_\tau^\alpha \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau + \frac{1}{2}\int_0^1 \theta(t) v^2 dx \\
\leq\ & \theta(0)\int_0^1 g^2(x)dx + \int_0^t\int_0^1 \theta'^2 v^2 dx d\tau + \int_0^t \|f(x,\tau)\|^2_{L^2(0,1)} d\tau \\
&+\frac{1}{2^m}\int_0^t \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau.
\end{aligned} \tag{27}$$

Now, since

$$\begin{aligned}
\int_0^t \partial_\tau^\alpha \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau &= D^{\alpha-1}\|\mathcal{I}_x^m v_t\|^2_{L^2(0,1)} \\
&\quad -\frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|\mathcal{I}_x^m h\|^2_{L^2(0,1)},
\end{aligned} \tag{28}$$

evoking conditions (17) and using (28), we infer from (26) that

$$\begin{aligned}
&D^{\alpha-1}\|\mathcal{I}_x^m v_t\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} \\
\leq\ & \eta\left(\|h\|^2_{L^2(0,1)} + \|g\|^2_{L^2(0,1)} + \int_0^t \|f\|^2_{L^2(0,1)} d\tau \right.\\
&\left. + \int_0^t \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau + \int_0^t\int_0^1 v^2(x,\tau) dx d\tau\right),
\end{aligned} \tag{29}$$

where

$$\eta = \frac{\max\left(1, c_1, c_3^2, 2^{-m}, \frac{T^{1-\alpha}2^{-m}}{(1-\alpha)\Gamma(1-\alpha)}\right)}{\min(1, c_2)}. \tag{30}$$

If, in Lemma 3, we set

$$\varphi(t) = \int_0^t\int_0^1 v^2(x,\tau)dx d\tau,\ \varphi'(t) = \|v\|^2_{L^2(0,1)},\ \text{and } \varphi(0) = 0, \tag{31}$$

then it yields

$$\begin{aligned}
\int_0^t\int_0^1 v^2(x,\tau)dx d\tau &\leq Te^{\eta T}\eta\left(\int_0^t \|f(x,\tau)\|^2_{L^2(0,1)} d\tau + \int_0^t \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau\right.\\
&\quad \left.+\|g(x)\|^2_{L^2(0,1)} + \|h(x)\|^2_{L^2(0,1)}\right).
\end{aligned} \tag{32}$$

Consequently, (30) transforms to

$$\begin{aligned} & D^{\alpha-1}\|\mathcal{I}_x^m v_t\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} \\ \leq\ & \delta\left(\int\limits_0^t \|f(x,\tau)\|^2_{L^2(0,1)} d\tau + \int\limits_0^t \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau \right. \\ & \left. + \|g(x)\|^2_{L^2(0,1)} + \|h(x)\|^2_{L^2(0,1)}\right), \end{aligned} \tag{33}$$

where

$$\delta = \max\left\{\eta, \eta^2 T e^{\eta}\right\}. \tag{34}$$

Now, by dropping the second term on the left-hand side of (33) then setting $\mathcal{Z}(t) = \int\limits_0^t \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau$, $\partial_t^\alpha \mathcal{Z}(t) = D^{\alpha-1}\|\mathcal{I}_x^m v_t\|^2_{L^2(0,1)}$ in Lemma 4, we obtain

$$\begin{aligned} & \int\limits_0^t \|\mathcal{I}_x^m v_\tau\|^2_{L^2(0,1)} d\tau \\ \leq\ & \delta\Gamma(\alpha)E_{\alpha,\alpha}\left(\delta t^\alpha\right)\left(D_t^{-\alpha-1}\|f\|^2_{L^2(0,1)} + \frac{T}{\alpha\Gamma(\alpha)}\|h\|^2_{L^2(0,1)} + \frac{T}{\alpha\Gamma(\alpha)}\|g\|^2_{L^2(0,1)}\right) \\ \leq\ & \delta\Gamma(\alpha)E_{\alpha,\alpha}\left(\delta t^\alpha\right)\max\left\{1, \frac{T}{\alpha\Gamma(\alpha)}\right\} \\ & \times\left(D_t^{-\alpha-1}\|f\|^2_{L^2(0,1)} + \|h\|^2_{L^2(0,1)} + \|g\|^2_{L^2(0,1)}\right). \end{aligned} \tag{35}$$

Combination of (33) and (36) leads to

$$\begin{aligned} & D^{\alpha-1}\|\mathcal{I}_x^m v_t\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} \\ \leq\ & \rho\left(\int\limits_0^t \|f(x,\tau)\|^2_{L^2(0,1)} d\tau + D_t^{-\alpha-1}\|f\|^2_{L^2(0,1)} \right. \\ & \left. + \|h\|^2_{L^2(0,1)} + \|g\|^2_{L^2(0,1)}\right), \end{aligned} \tag{36}$$

where

$$\rho = \delta\max\left(1, \delta\Gamma(\alpha)E_{\alpha,\alpha}\left(\delta t^\alpha\right)\max\left\{1, \frac{T}{\alpha\Gamma(\alpha)}\right\}\right). \tag{37}$$

It is obvious that

$$\begin{aligned} D_t^{-\alpha-1}\|f\|^2_{L^2(0,1)} &\leq \frac{t^\alpha}{\Gamma(1+\alpha)}\int_0^t \|f\|^2_{L^2(0,1)}\, dt, \\ &\leq \frac{T^\alpha}{\Gamma(1+\alpha)}\int_0^T \|f\|^2_{L^2(0,1)}\, dt. \end{aligned} \tag{38}$$

Then, it follows from (37) and (38) that

$$\begin{aligned} & D^{\alpha-1}\|\mathcal{I}_x^m v_t\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} \\ \leq\ & \mathcal{K}\left(\|f\|^2_{L^2(Q)} + \|h\|^2_{L^2(0,1)} + \|g\|^2_{L^2(0,1)}\right), \end{aligned} \tag{39}$$

where

$$\mathcal{K} = \rho\left(1 + \frac{T^\alpha}{\Gamma(1+\alpha)}\right).$$

Observe that the right-hand side of (39) is independent of the variable t, so we are allowed to take the least upper bound of the left-hand side with respect to t over $[0, T]$, and the a priori estimate (18) then follows and from which we deduce the uniqueness and continuous dependence of the solution on the input data of problem (1). □

5. Existence of Solution

In this section, we prove the main result concerning the existence of solution of problem (1) The a priori estimate (18) shows that the unbounded operator $\mathcal{M}$ has an inverse $\mathcal{M}^{-1} : R(\mathcal{M}) \to \mathcal{B}$. Since $R(\mathcal{M})$ is a subset of $\mathcal{Y}$, we therefore can construct its closure $\overline{\mathcal{M}}$ in a manner that the estimate (18) holds for this extension and $R(\overline{\mathcal{M}})$ coincides with the whole space $\mathcal{B}$. Hence, the following:

Corollary 1. *The operator $\mathcal{M} : \mathcal{B} \to \mathcal{Y}$ admits a closure (the proof is similar to that in [14]).*
Estimate (18) can be then extended to

$$\begin{aligned} & \sup_{0 \le t \le T} \left(D_t^{\alpha-1} \|\mathcal{I}_x^m v_t(x,t)\|^2_{L^2(0,1)} + \int_0^1 v^2(x,t)dx \right) \\ \le \; & \mathcal{K} \left(\|g\|^2_{L^2(0,1)} + \|h\|^2_{L^2(0,1)} + \|f\|^2_{L^2(Q)} \right) \end{aligned} \tag{40}$$

for all $v \in D(\overline{\mathcal{M}})$.

It follows from (40) that the strong solution of problem (1) is unique, that is, $\overline{\mathcal{M}}v = \mathcal{Y}$. We also deduce from estimate (40) the following:

Corollary 2. *$R(\overline{\mathcal{M}})$ is a closed subset in $\mathcal{Y}$ and $R(\mathcal{M})$ $=R(\overline{\mathcal{M}})$ and $\overline{\mathcal{M}}^{-1} = \overline{\mathcal{M}^{-1}}$.*

We are now ready to give the result of existence of the solution of problem (1).

Theorem 2. *Suppose that conditions of Theorem 4.1 are satisfied. Then, for all $\mathcal{W} = (f, g, h) \in \mathcal{Y}$, there exists a unique strong solution $v = \overline{\mathcal{M}}^{-1}\mathcal{W} = \overline{\mathcal{M}^{-1}}\mathcal{W}$ of problem (1).*

Proof. Estimate (40) asserts that, if a strong solution of (1) exists, it is unique and depends continuously on the data. Corollary 2 says that, in order to prove that problem (1) admits a strong solution for any $\mathcal{W} = (f, g, h) \in \mathcal{Y}$, it suffices to show that the closure of the range of the operator $\mathcal{M}$ is dense in $\mathcal{Y}$. To establish the existence of the strong solution of problem (1), we use a density argument. That is, we show that the range $R(\mathcal{M})$ of the operator $\mathcal{M}$ is dense in the space $\mathcal{Y}$ for every element v in the Banach space $\mathcal{B}$. For this, we consider the following special case of density. □

Theorem 3. *Suppose that conditions of Theorem 1 are satisfied. If for all functions $v \in \mathcal{D}(\mathcal{M})$ such that $l_1 v = v(x,0) = 0$, $l_2 v = v_t(x,0) = 0$ and for some function $\psi \in L^2(Q)$, we have*

$$\int_0^T (\mathcal{L}v, \psi)_{L^2(0,1)}\, dt = 0, \tag{41}$$

then ψ vanishes a.e in Q.

Proof. Identity (40) is equivalent to

$$\int_0^T \left(\partial_t^{\alpha+1} v + (-1)^m \theta(t) \frac{\partial^{2m} v}{\partial x^{2m}}, \psi \right)_{L^2(0,1)} dt = 0. \tag{42}$$

Assume that a function $\sigma(x,t)$ verifies conditions boundary and initial conditions in (1) and such that σ, σ_x, $\mathcal{I}_t\sigma$, $\mathcal{I}_t\mathcal{I}_x^{2m}\sigma$, $\mathcal{I}_t^2\sigma$ and $\partial_t^{\beta+1}\sigma \in L^2(Q)$, we then set

$$v(x,t) = \mathcal{I}_t^2\sigma = \int_0^t \int_0^s \sigma(x,z)dzds. \tag{43}$$

Equation (42) then becomes

$$\int_0^T \left(\partial_t^{\alpha+1}\mathcal{I}_t^2\sigma + (-1)^m\theta(t).\frac{\partial^{2m}}{\partial x^{2m}}(\mathcal{I}_t^2\sigma), \psi\right)_{L^2(0,1)} dt = 0. \tag{44}$$

We now introduce the function

$$\psi(x,t) = \mathcal{I}_t\sigma + (-1)^m\mathcal{I}_x^{2m}\mathcal{I}_t\sigma. \tag{45}$$

Equation (44) then reduces to

$$\begin{aligned} &\int_0^T \left(\partial_t^{\alpha+1}\mathcal{I}_t^2\sigma, \mathcal{I}_t\sigma\right)_{L^2(0,1)} dt + \int_0^T \left(\partial_t^{\alpha+1}\mathcal{I}_t^2\sigma, (-1)^m\mathcal{I}_x^{2m}\mathcal{I}_t\sigma\right)_{L^2(0,1)} dt \\ &+ \int_0^T \left((-1)^m\theta(t).\frac{\partial^{2m}}{\partial x^{2m}}(\mathcal{I}_t^2\sigma), \mathcal{I}_t\sigma\right)_{L^2(0,1)} dt \\ &+ \int_0^T \left(\theta(t).\frac{\partial^{2m}}{\partial x^{2m}}(\mathcal{I}_t^2\sigma), \mathcal{I}_x^{2m}\mathcal{I}_t\sigma\right)_{L^2(0,1)} dt \\ = \;& 0. \end{aligned} \tag{46}$$

Recall that the function σ satisfies boundary conditions in (1) and then, computing the inner products in (45), one has

$$\begin{aligned} \left(\partial_t^{\alpha+1}\mathcal{I}_t^2\sigma, \mathcal{I}_t\sigma\right)_{L^2(0,1)} &= \left(\partial_t^{\alpha}\mathcal{I}_t\sigma, \mathcal{I}_t\sigma\right)_{L^2(0,1)}, \\ &\geq \frac{1}{2}\partial_t^{\alpha}\|\mathcal{I}_t\sigma\|^2_{L^2(0,1)}, \end{aligned} \tag{47}$$

$$\begin{aligned} &\left(\partial_t^{\alpha+1}\mathcal{I}_t^2\sigma, (-1)^m\mathcal{I}_x^{2m}\mathcal{I}_t\sigma\right)_{L^2(0,1)} = \left(\partial_t^{\alpha}\mathcal{I}_x^m\mathcal{I}_t\sigma, \mathcal{I}_x^m\mathcal{I}_t\sigma\right)_{L^2(0,1)}, \\ \geq \;& \frac{1}{2}\partial_t^{\alpha}\|\mathcal{I}_x^m\mathcal{I}_t\sigma\|^2_{L^2(0,1)}, \end{aligned} \tag{48}$$

$$\begin{aligned} &\left((-1)^m\theta(t).\frac{\partial^{2m}}{\partial x^{2m}}(\mathcal{I}_t^2\sigma), \mathcal{I}_t\sigma\right)_{L^2(0,1)} \\ = \;& \left(\theta(t)\frac{\partial^m}{\partial x^m}(\mathcal{I}_t^2\sigma), \frac{\partial^m}{\partial x^m}(\mathcal{I}_t P)\right)_{L^2(0,1)} \\ = \;& \frac{1}{2}\frac{d}{dt}\int_0^1 \theta(t)\left(\frac{\partial^m}{\partial x^m}(\mathcal{I}_t^2\sigma)\right)^2 dx - \frac{1}{2}\int_0^1 \theta'(t)\left(\frac{\partial^m}{\partial x^m}(\mathcal{I}_t^2\sigma)\right)^2 dx, \end{aligned} \tag{49}$$

$$\begin{aligned}
&\left(\theta(t).\frac{\partial^{2m}}{\partial x^{2m}}(\mathcal{I}_t^2\sigma),\mathcal{I}_x^{2m}\mathcal{I}_t\sigma\right)_{L^2(0,1)} \\
= &\left(\theta(t)(\mathcal{I}_t^2\sigma),\mathcal{I}_t\sigma\right)_{L^2(0,1)} = \frac{1}{2}\frac{d}{dt}\int_0^1 \theta(t)(\mathcal{I}_t^2\sigma)^2 dx \qquad (50)\\
&-\frac{1}{2}\int_0^1 \theta'(t)(\mathcal{I}_t^2\sigma)^2 dx.
\end{aligned}$$

Evoking (47)–(51), replace t by τ, integrating with respect to τ from zero to t and using conditions (17), we obtain

$$\begin{aligned}
&D_t^{\alpha-1}\|\mathcal{I}_t\sigma\|^2_{L^2(0,1)} + D_t^{\alpha-1}\|\mathcal{I}_x^m\mathcal{I}_t\sigma\|^2_{L^2(0,1)} + \int_0^1\left(\frac{\partial^m}{\partial x^m}(\mathcal{I}_t^2\sigma)\right)^2 dx \\
&+\int_0^1 (\mathcal{I}_t^2\sigma)^2 dx \qquad (51)\\
\leq\ & \frac{c_3}{\min(1,c_1)}\left(\int_0^t\int_0^1\left(\frac{\partial^m}{\partial x^m}(\mathcal{I}_\tau^2\sigma)\right)^2 dx d\tau + \int_0^t\int_0^1 (\mathcal{I}_\tau^2\sigma)^2 dx d\tau\right).
\end{aligned}$$

By dropping the first two terms on the left-hand side of (50) and applying Gronwall's Lemma 5, by setting $Z_1(t) = 0$, $Z_3(t) = 0$ and

$$Z_2(t) = \int_0^1\left(\frac{\partial^m}{\partial x^m}(\mathcal{I}_t^2\sigma)\right)^2 dx + \int_0^1(\mathcal{I}_t^2\sigma)^2 dx,$$

we have

$$\int_0^1\left(\frac{\partial^m}{\partial x^m}(\mathcal{I}_t^2\sigma)\right)^2 dx + \int_0^1(\mathcal{I}_t^2\sigma)^2 dx \leq 0 \qquad (52)$$

for all $t \in [0,T]$. Hence, $\psi = 0$ a.e in Q. □

To complete the proof of Theorem 2, and prove the density ($\overline{R(\mathcal{M})} = \mathcal{Y}$) in a general case, suppose that, for some element $(F_1,\theta_1,\theta_2) \in R(\mathcal{M})^{\perp}$, we have

$$\int_0^T (\mathcal{L}v, F_1)_{L^2(0,1)}\, ds + (l_1 v,\theta_1)_{L^2(0,1)} + (l_2 v,\theta_2)_{L^2(0,1)} = 0, \qquad (53)$$

and then we prove that $F_1 = 0$, $\theta_1 = 0$, $\theta_2 = 0$. If we put $v \in D(\mathcal{M})$ satisfying conditions $l_1 v = v(x,0) = 0$ and $l_2 v = v_t(x,0) = 0$ into (53), we obtain

$$\int_0^T (\mathcal{L}v, F_1)_{L^2(0,1)}\, ds = 0,\ \forall v \in D(\mathcal{M}). \qquad (54)$$

By Theorem 3, Equation (54) implies that $F_1 = 0$ a.e in Q. Then, (53) becomes

$$(l_1 v,\theta_1)_{L^2(0,1)} + (l_2 v,\theta_2)_{L^2(0,1)} = 0\ \ \forall\theta \in D(\mathcal{M}). \qquad (55)$$

Since the range of the trace operator l_1 and l_2 is dense in $L^2(0,1)$, it follows then from (55) that $\theta_1 = 0$, $\theta_2 = 0$. This ends the proof of Theorem 2.

6. Conclusions

The existence and uniqueness of a generalized solution for a higher order fractional diffusion wave equation in Caputo sense subject to initial and weighted integral boundary conditions are established. It is found that the method of energy inequalities is successfully applied to obtaining a priori bounds for the solution of fractional initial boundary value problems of higher order with nonlocal constraints as in the classical case. The obtained results will contribute to the development of the functional analysis method and enrich the existing non-extensive literature on the non local fractional mixed problems in the Caputo sense.

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