Exercises
Theory of statistics 2
STAT 419

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Exercises 1 - Chapter 1 :Introduction

Distribution related :

- Let $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_1$.
- Let $Z_1, Z_2, ..., Z_n$ i.i.d random variables from $N(0,1)$ .Then $\sum_{i=1}^{n} Z_i^2 \sim \chi^2_n$.
- Let $X_1, X_2, ..., X_n$ i.i.d random variables from $N(\mu, \sigma^2)$:

  (i) If μ is known . Then $\sum_{i=1}^{n} (\frac{X_i - \mu}{\sigma})^2 \sim \chi^2_n$.

  (ii) If μ is Unknown . Then $\sum_{i=1}^{n} (\frac{X_i - \bar{X}}{\sigma})^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.
- Let $X \sim \chi^2_n, Y \sim \chi^2_m$. If X, Y independent then $X + Y \sim \chi^2_{n+m}$.
- Let $Z \sim N(0,1)$, $U \sim \chi^2_n$ and if the Z ,U independent , then $\frac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$.
- Let $U \sim \chi^2_n$ and $V \sim \chi^2_m$ are independent chi-square variables with n and m degrees of freedom respectively. Then $\frac{\frac{U}{n}}{\frac{V}{m}} \sim F_{n,m}$.
- If $X \sim Gamma(n, \theta)$,then $2\theta X \sim \chi^2_{2n}$.

Example 1:
Proof
If $X_1, X_2, ..., X_n \sim Exp(\theta)$ iid (independent and identical distributed) , then $\sum_{i=1}^{n} X_i \sim Gamma(n, \theta)$.

Solution 1:
By using "moment generating function (MGF)" :

$$X \sim Exp(\theta) \Rightarrow f(x) = \theta e^{-\theta x}, M_x(t) = \frac{\theta}{\theta - t}$$

Let $Y = \sum_{i=1}^{n} X_i$

$$\begin{aligned} M_Y(t) &= E(e^{tY}) \\ &= E(e^{t(x_1+x_2+....+x_n)}) \\ &= E(e^{tx_1}e^{tx_2}....e^{tx_n}) \\ &= E(e^{tx_1})E(e^{tx_2})....E(e^{tx_n}) \\ &= M_{x_1}(t).M_{x_2}(t).....M_{x_n}(t) \\ &= \frac{\theta}{\theta - t}.\frac{\theta}{\theta - t}........\frac{\theta}{\theta - t} \\ &= (\frac{\theta}{\theta - t})^n \end{aligned}$$

So, $Y \sim Gamma(n, \theta)$.

Example 2:
Proof
If $X \sim Gamma(n, \theta)$,then $2\theta X \sim \chi^2_{2n}$.

Solution 2:
By using "transformation method " :
The random variable X have the gamma distribution with probability density function:

$$X \sim Gamma(n, \theta) \Rightarrow f(x) = \frac{\theta^n}{\Gamma(n)} x^{n-1} e^{-x\theta}$$

- $Y = 2\theta x \implies x = \frac{Y}{2\theta}, \quad g^{-1}(Y) = \frac{Y}{2\theta}$
- $\frac{d}{dy} g^{-1}(Y) = \frac{1}{2\theta}$

By the transformation technique, the probability density function of Y is:

- $f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy} g^{-1}(y)|$

$$\begin{aligned} f_Y(y) &= \frac{\theta^n}{\Gamma(n)}(\frac{y}{2\theta})^{n-1} e^{-\frac{y}{2\theta}\theta} \frac{1}{2\theta} \\ &= \frac{1}{2^n \Gamma(n)} y^{n-1} e^{-\frac{y}{2}} \end{aligned}$$

So, $Y \sim \chi^2_{2n}$.

Example 3 :
Proof : If $X \sim U(0,1)$, then $T = -\log X \sim Exp(1)$.

Solution 3:

$$X \sim U(0,1) \Rightarrow f(x) = 1$$

- $t = -\log x \implies g^{-1}(t) = e^{-t}$
- $\frac{d}{dt} g^{-1}(t) = -e^{-t}$
- $f_T(t) = f_X(g^{-1}(t))|\frac{d}{dt} g^{-1}(t)|$

$$\begin{aligned} f_T(t) &= 1.e^{-t} \\ &= e^{-t} \end{aligned}$$

So, $t \sim Exp(\theta = 1)$.

Order Statistic :
Theorem : Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with density function $f(x)$. Then the probability density function of the r^{th} order statistic, $X_{(r)}$, is

$$g_r(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x)(1 - F(x))^{n-r},$$

where $F(x)$ denotes the cdf of $f(x)$.

Example 4:
Let $X_{(1)} < X_{(2)} < .... < X_{(6)}$ be the order statistics from a random sample of size 6 from a distribution with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1. \\ 0 & \text{otherwise.} \end{cases}$$

Find density function of $X_{(6)}$, and what is the expected value of $X_{(6)}$?

Solution 4:

$$f(x) = 2x$$

$$\begin{aligned} F(x) &= \int_0^x 2t dt \\ &= x^2 \end{aligned}$$

The density function of X_6 is given by :

$$\begin{aligned} g_r(x) &= \frac{n!}{(r-1)!(n-r)!}F(x)^{r-1}f(x)(1-F(x))^{n-r} \\ g_6(x) &= \frac{6!}{(6-1)!(6-6)!}(x^2)^{6-1}2x(1-x^2)^{6-6} \\ &= \frac{6!}{5!0!}(x^2)^5 2x \\ &= 12x^{11}. \end{aligned}$$

Hence, the expected value of X_6 is:

$$\begin{aligned} E(X_6) &= \int_0^1 x g_6(x)dx \\ &= \int_0^1 x12x^{11}dx \\ &= \frac{12}{13}[x^{13}]_0^1 \\ &= \frac{12}{13}. \end{aligned}$$

Example 5 : class activity

Let $Y_{(1)} < Y_{(2)} < ... < Y_{(6)}$ be the order statistics associated with $n = 6$ independent observations each from the distribution with probability density function:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 < x < 2. \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability density function of the first order statistics?

Solution 5:

Applying the theorem with $n = 6$ and $r = 1$, the p.d.f. of Y_1 is:

$$\begin{aligned} g_r(y) &= \frac{n!}{(r-1)!(n-r)!}F(x)^{r-1}f(x)(1-F(x))^{n-r} \\ g_1(y) &= 3y(1-\frac{y^2}{4})^5. \end{aligned}$$

Example 6: Homework

Choose the correct answer :

1-If $Z_1, Z_2, ..., Z_5$ i.i.d random variables from $N(0,1)$ .Then $\sum_{i=1}^{5} Z_i^2$ follow :

(A) χ_n^2 (B) t_4 (C) χ_5^2 (D) $N(0,1)$

2-If $X \sim \chi_1^2$, $Y \sim \chi_2^2$ and $W \sim \chi_4^2$. If X, Y, W independent then $X + Y + W$ follow :

(A) χ_n^6 (B) t_7 (C) $F_{3,4}$ (D) χ_7^2

3-If $X \sim Gamma(1, \beta)$, then also X has .....:

(A) Chi-squared distribution with β degrees of freedom
(B) Chi-squared distribution with 1 degrees of freedom
(C) Exponential distribution(β)
(D) Exponential distribution(1)

4-Let $Y_{(1)} < Y_{(2)} < ... < Y_{(6)}$ be the order statistics associated with $n = 6$ independent observations each from the distribution with probability density function:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 < x < 2. \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability density function of the fourth order statistics?

Solution 6:

1- C 2- D 3- C

4-

$$\begin{aligned} g_r(y) &= \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x)(1 - F(x))^{n-r} \\ g_4(y) &= \frac{15}{32} y^7 (1 - \frac{y^2}{4})^2. \end{aligned}$$