Exercises 2
Chapter 2 :Pivotal Quantity PQ- Confidence Interval (C.I) by PQ

Definition PQ : Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population **X** with probability density function $f(x;\theta)$, where θ is an unknown parameter. A pivotal quantity Q is a function of $X_1, X_2, ..., X_n$ and θ whose probability distribution is independent(does not depend) of the parameter θ.

Example 1 :

1- If $X_1, ., X_n$ be a random sample from $N(\mu, \sigma^2)$,σ^2 is known ,Then $Q = \frac{\overline{X}-\mu}{\sigma \div \sqrt{n}} \sim N(0,1)$, is Q Pivotal Quantity for μ , Why ?

Yes is a pivotal quantity(PQ), since it is a function of $X_1, X_2, ..., X_n$ and μ and its distribution free of the parameter μ .

2- If $X_1, ., X_n$ be a random sample from $N(\mu, \sigma^2)$,σ^2 is Unknown ,Then $Q = \frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$, is Q Pivotal Quantity for μ , Why ?

Yes is a pivotal quantity(PQ) , since it is a function of $X_1, X_2, ..., X_n$ and μ and its distribution free of the parameter μ .

Example 2:

Let $X_1, X_2, ..., X_n$ a random sample from $N(\theta, 9)$, then :

1- $Q = \bar{X} - \theta$

2- $Q = \frac{\bar{X}-\theta}{\frac{3}{\sqrt{n}}}$

3- $Q = \frac{\bar{X}}{\theta}$

Are Q Pivotal Quantity for θ and Why ?

Solution 3 :

1- Yes , is a pivotal quantity(PQ) since it is a function of $X_1, X_2, ..., X_n$ and θ and its distribution free of the parameter θ .
($Q = \overline{X} - \theta$ is normally distributed with mean 0 and variance $\frac{9}{n}$).

2-Yes , is a pivotal quantity(PQ) since it is a function of $X_1, X_2, ..., X_n$ and θ and its distribution free of the parameter θ .
($Q = \frac{\overline{X}-\theta}{\frac{3}{\sqrt{n}}}$ has standard normal distribution).

3- No , is not a pivotal quantity(PQ) since $Q = \frac{\overline{X}}{\theta}$ is normally distributed with mean 1 and variance $\frac{9}{n\theta^2}$, which depends on θ .

Confidence Interval (C.I) by PQ :

Our aim is to utilize a pivotal quantity to obtain a confidence interval :
1- $P(q_1 < Q(\underline{X}, \theta) < q_2) = P(T_1(\underline{X}) < \tau(\theta) < T_2(\underline{X})) = 1 - \alpha$.
2-The length $L = T_2(\underline{X}) - T_1(\underline{X})$ must be minimum .

Example 3:

**Let X be random variable with normal distribution $N(\mu, \sigma^2)$, σ^2 is known :
let $X = (X_1, ., X_n)$ be n copies of X . Find $(1-\alpha)100$ Confidence interval for μ ?
Use $Q = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim g(q) = N(0,1)$ as P.Q**

Solution 3 :

Step 1:

$$P(q_1 < Q(\underline{X}, \theta) < q_2) = 1 - \alpha \quad \Longleftrightarrow \quad \int_{q_1}^{q_2} g(q)dq = 1 - \alpha \tag{1}$$

$$\begin{aligned} P(q_1 < Q(\underline{X},\theta) < q_2) &= P(q_1 < \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < q_2) \\ &= P(q_1\frac{\sigma}{\sqrt{n}} < \overline{X}-\mu < q_2\frac{\sigma}{\sqrt{n}}) \\ &= P(q_1\frac{\sigma}{\sqrt{n}} - \overline{X} < -\mu < q_2\frac{\sigma}{\sqrt{n}} - \overline{X}) \\ &= P(\overline{X} - q_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} - q_1\frac{\sigma}{\sqrt{n}}) \end{aligned}$$

Step 2:
The length of confidence interval is given by :

$$\begin{aligned} L &= T_2(\underline{X}) - T_1(\underline{X}) \\ &= [\overline{X} - q_1\frac{\sigma}{\sqrt{n}}] - [\overline{X} - q_2\frac{\sigma}{\sqrt{n}}] \\ &= \frac{\sigma}{\sqrt{n}}(q_2 - q_1) \qquad \textbf{must be minimum} \end{aligned} \tag{2}$$

Now , Differentiate (1)**with respect to** q_1 **. We get :**

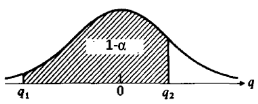
$$\frac{d}{dq_1}[\int_{q_1}^{q_2} g(q)dq = 1-\alpha \quad]$$

$$g(q_2)\frac{dq_2}{dq_1} - g(q_1) = 0 \qquad >> \qquad \frac{dq_2}{dq_1} = \frac{g(q_1)}{g(q_2)} \tag{3}$$

let us differentiate L **with respect to** q_1 **, we get :**

$$\begin{aligned} \frac{dL}{dq_1} &= \frac{d}{dq_1}[\frac{\sigma}{\sqrt{n}}(q_2 - q_1)] \\ &= \frac{\sigma}{\sqrt{n}}(\frac{dq_2}{dq_1} - 1) \qquad \textbf{from (3) :} \frac{dq_2}{dq_1} = \frac{g(q_1)}{g(q_2)} \\ &= \frac{\sigma}{\sqrt{n}}(\frac{g(q_1)}{g(q_2)} - 1) \qquad \textbf{g(q) follows standard normal distribution} \\ &= \frac{\sigma}{\sqrt{n}}(\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{q_1^2}{2}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{q_2^2}{2}}} - 1) \\ &= \frac{\sigma}{\sqrt{n}}(e^{-\frac{1}{2}(q_1^2 - q_2^2)} - 1) \end{aligned}$$

Thus $\frac{dL}{dq_1} = 0$ **if and only if** $q_1 = q_2$ **or** $q_1 = -q_2$ **.**
Since $q_1 < q_2$ **(from the first step) , then the minimum of the function L is obtained on** $q_1 = -q_2$ **. And it follows that** $q_2 = Z_{1-\frac{\alpha}{2}}$ **.**



C.I for μ **:**

$$\mu \in (\overline{X} - q_2\frac{\sigma}{\sqrt{n}}, \overline{X} - q_1\frac{\sigma}{\sqrt{n}})$$

$$\mu \in (\overline{X} \pm Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$$