Exercises To Introduction to Partial Differential Equations

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1 Basic Concepts

Exercise 1

1. Give the order of each of the following PDEs:
   (a) $u_{xx} + u_{yy} = 0$
   (b) $u_{xxx} + u_{xy} + a(x)u_y + \log u = f(x, y)$
   (c) $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x, y)$
   (d) $uu_{xx} + u_{yy}^2 + e^u = 0$
   (e) $u_x + cu_y = d$
2. Which of the following PDEs is linear? quasilinear? nonlinear? If it is linear, state whether it is homogeneous or not.
   (a) $u_{xx} + u_{yy} - 2u = x^2$
   (b) $u_{xy} = u$
   (c) $uu_x + xu_y = 0$
   (d) $u_x^2 + \log u = 2xy$
   (e) $u_{xx} - 2u_{xy} + u_{yy} = \cos x$
   (f) $u_x\,(1 + u_y) = u_{xx}$
   (g) $(\sin u_x)\, u_x + u_y = e^x$
   (h) $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
   (i) $u_x + u_x u_y - u_{xy} = 0$
3. Eliminate the arbitrary functions to obtain a differential equation of lowest order:
   (a) $u = e^{-x} f(x - 2y)$
   (b) $u = xf(x + y) + g(x + y)$

(c) $u = xf(y+cx) + yg(y+cx)$

4. Eliminate the arbitrary constants to obtain a differential equation of lowest order:

   (a) $u = c_1 \cos x + c_2 \sin y$

   (b) $u = c_1(x+y) + c_2(x-y) + c_1 c_2 t + c_3$

5. Find the differential equation of all sphere of radius λ and having centre on xy-plane?

6. Show that
$$u = F(xy) + xG(\frac{y}{x})$$
is the general solution of
$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

7. Find the general solution of
$$u_{xy} + u_y = 0.$$
(Hint:Let $v = u_y$)

2 First-Order Equations

Exercise 2:

1. Solve the following first-order partial differential equations:

   (a) $u_{xx} = 6x$

   (b) $5u_x + 4u_y + u = x^3 + 1 + 2e^{3y}$

   (c) $5u_x + u_y = y.$

   (d) $(y-x)u_x + (y+x)u_y = \frac{x^2+y^2}{u}.$

   (e) $2yu_x + (3x^2-1)u_y = 0.$

   (f) $(u^2 - 2yu - y^2)u_x + (xy + xu)u_y = xy - xu.$

   (g) $xu_x - 7yu_y = x^2 y.$

   (h) $u_x + u_y + u_z = 0.$

2. Find the solution of Cauchy problem:

   (a) $y^{-1}u_x + u_y = u^2,\ u(x,1) = x^2.$

   (b) $xu_x + yu_y = x^2 - y,\ u(1,y) = y.$

   (c) $u_x + 3u_y = u^2,\ u(x,0) = h(x)$, where $h(x)$ is a smooth function and not equal to zero.

3. Find the integral surface of the given differential equation which passes through the curve Γ:

   (a) $(y+xu)u_x + (x+yu)u_y = u^2 - 1, \quad \Gamma : x = t, y = 1, u = t^2.$

   (b) $u_x + \sqrt{u}u_y = 0, \quad \Gamma : u(x,0) = x^2 + 1.$

   (c) $u(u_x - u_y) = y - x, \quad \Gamma : u(1,y) = y^2.$

   (d) $e^u(u_x + x^2 u_y) = \frac{1}{1+x}, \quad \Gamma : x = 0, y = t, u = t.$

3 Second-Order Equations

Exercise 3:

1. Classify each equation. Reduce to canonical form.

   (a) $u_{xx} + u_{xy} - 6u_{yy} = 0.$

   (b) $u_{xy} - u_y = \cos y.$

   (c) $u_{xx} - 2u_{xy} + u_{yy} = 1.$

   (d) $x^2 u_{xx} - y^2 u_{yy} = (x+y)xy.$

2. Find the general solution of following PDEs

   (a) $u_{yy} + u = e^{x+y}.$

   (b) $(x-y)u_{xy} - u_x + u_y = 0.$ $\Big[\text{Hint: use } v = (x-y)u\Big].$

   (c) $u_{xx} - 9u_{yy} = x^2 + \sin y + 2.$

3. Find the solution of Cauchy problem:

   (a) $u_{xx} - u_{yy} = 0$

   $$u(x,x) = 0, \quad \textbf{and} \quad u_x(x,x) = 1.$$

   (b) $u_{xy} + u_y = y$

   $$u(x,x) = x^2, \quad \textbf{and} \quad u_x(x,x) = 0.$$

4 Sturm/Liouville Theory and Orthogonal Functions

Exercise 4:

1. Determine the coefficients a_i and b_i in the linear combination

   $$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$$

   which give the best approximation in $\mathcal{L}^2(-\pi,\pi)$ of $f(x) = |x|, -\pi \le x \le \pi.$

2. Put the following operators in the form

$$p\frac{d^2}{dx^2} + p'\frac{d}{dx} + r,$$

where $p > 0$:

   (a) $x^2 \frac{d^2}{dx^2}$, $x > 0$,

   (b) $\frac{d^2}{dx^2} - x\frac{d}{dx}$, $x \in \mathbb{R}$,

   (c) $\frac{d^2}{dx^2} - x^2\frac{d}{dx}$, $x > 0$.

3. Put the equation $u'' + 2u' + \lambda u = 0$ in the standard form $Lu + \lambda\mu u = 0$ where the operator L is formally self-adjoint, then find the eigenvalues and eigenfunctions on $[0, 1]$ under the boundary conditions $u(0) = u(1) = 0$.

4. Find the eigenvalues and eigenfunctions for

$$u'' + \lambda u = 0, \quad a \leq x \leq b,$$

$$u(a) = u(b) = 0.$$

5 Laplace Equation

Exercise 5:

1. Find the solution of the following Dirichlet problem

   (a) $u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, 0 < y < 2\pi$

$$u(0, y) = u(\pi, y) = 0, \quad 0 < y < 2\pi$$

$$u(x, 0) = 0, u(x, 2\pi) = 1, \quad 0 < x < \pi$$

   (b) $u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b$

$$u(0, y) = u(a, y) = 0, \quad 0 < y < b$$

$$u(x, 0) = 0, u(x, b) = f(x), \quad 0 < x < a$$

   (c) $u_{xx} + u_{yy} = 0, \quad 0 < x < 2, 0 < y < 2$

$$u(0, y) = u(2, y) = 0, \quad 0 < y < 2$$

$$u(x, 0) = f(x), u(x, 2) = g(x), \quad 0 < x < 2$$

2. Find the solution of the following Neumann problem

   (a) $u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, 0 < y < \pi,$

   $$u_x(0, y) = u_x(\pi, y) = 0, \quad 0 < y < \pi$$

   $$u_y(x, 0) = u_y(x, \pi) = 0, \quad 0 < x < \pi.$$

   (b) $u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, 0 < y < \pi,$

   $$u_x(0, y) = u_x(\pi, y) = 0, \quad 0 < y < \pi$$
   $$u_y(x, 0) = 0, u_y(x, \pi) = \cos x, \quad 0 < x < \pi$$

3. Find the solution of the following mixed problem

   $$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, 0 < y < 1$$
   $$u_x(0, y) = u_x(\pi, y) = 0, \quad 0 < y < 1$$
   $$u(x, 0) = \cos^2 x, u_y(x, 1) = 0, \quad 0 < x < \pi$$

4. Find the solution in the polar coordinates:

   (a) $\Delta u = 0, \quad 0 \le r < 1, 0 < \theta < 2\pi$

   $$u(1, \theta) = \sin\theta - \sin 3\theta, \quad 0 \le \theta < 2\pi$$

   (b) $\Delta u = 0, \quad r > 0, 0 < \theta < \pi/2$

   $$u_\theta(r, 0) = 0, \quad 0 < r < 1,$$

   $$u(r, \pi/2) = \left(r^2 + 1\right)^2, \quad 0 < r < 1$$

   (c) $\Delta u = 0, \quad 1 < r < 2, 0 \le \theta < 2\pi$

   $$u(1, \theta) = 3, u(2, \theta) = 4, \quad 0 \le \theta < 2\pi$$

5. Find the solution of Laplace equation in $\mathbb{R}$, then find the solution of Dirichlet problem:

   $$u_{xx} + u_{yy} + u_{zz} = 0, \quad (x, y, z) \in (0, a) \times (0, b) \times (0, c)$$

   $$u(x, y, 0) = f(x, y), \quad (x, y) \in (0, a) \times (0, b)$$

   $$u(0, y, z) = u(a, y, c) = u(x, 0, z) = u(x, b, z) = u(x, y, c) = 0$$

6. Find u which satisfies the Laplace equation in the half of sphere

   $$0 \le r < 1, 0 \le \phi \le \pi/2$$

   if

   $$u(r, \pi/2) = 0 \quad \forall r \in [0, 1), \quad u(1, \phi) = 1 \quad \forall \phi \in [0, \pi/2],$$

6 Wave Equation

Exercise 6:

1. $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad$ for all $0 < x < 1$ and $t > 0$

$$u(0,t) = u(1,t) = 0 \qquad \text{for all } t > 0$$

$$u(x,0) = x(1-x) \qquad \text{for all } 0 < x < 1$$

$$u_t(x,0) = 0 \qquad \text{for all } 0 < x < 1$$

2. $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad$ for all $0 < x < 1$ and $t > 0$

$$u(0,t) = u(1,t) = 0 \qquad \text{for all } t > 0$$

$$u(x,0) = \sin(5\pi x) + 2\sin(7\pi x) \qquad \text{for all } 0 < x < 1$$

$$u_t(x,0) = 0 \qquad \text{for all } 0 < x < 1$$

3. $\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \tanh(x) & x \in \mathbb{R} \\ u_t(x,0) = \arctan(x) & x \in \mathbb{R} \end{cases}$

4. Find the solution of

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad x, t \in \mathbb{R} \\ u(x,0) = e^{-x^2}, \quad u_t(x,0) = 2axe^{-x^2} \end{cases}$$

5. $u_{tt} = 25u_{xx} \quad 0 < x < 2$

$$u(x,0) = x^2(2-x), \quad u_t(x,0) = x^2, \quad u(0,t) = u(2,t) = 0$$

7 Heat Equation

Exercise 7:

1. $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \; t > 0.$

$$u(x,0) = 6\sin\left(\frac{\pi x}{L}\right)$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

2. $\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < L, \;\; t > 0$

$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

$$u(x,0) = f(x)$$

3. $u_t = k u_{xx}, \quad x > 0, \;\; t > 0.$