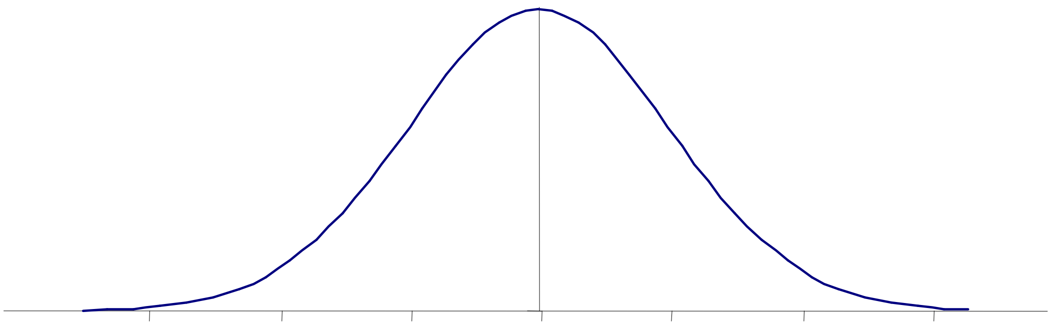
Confidence Interval and Hypothesis Testing: Exercises and Solutions



You can use the graphical representation of the normal distribution to solve the problems.

Exercise 1: Confidence Interval

A machine is set up such that the average content of juice per bottle equals μ.
A sample of 100 bottles yields an average content of 48cl.
Calculate a 90% and a 95% confidence interval for the average content.

Assume that the population standard deviation $\sigma = 5cl$.

Exercise 2: Sample size

What sample size is required to estimate the average contents
to within 0.5cl at the 95% confidence level? (= + or – 0.5 cl)

Assume that the population standard deviation $\sigma = 5cl$.

Exercise 3: Hypothesis Testing

A machine is set up such that the average content of juice per bottle equals μ.
A sample of 36 bottles yields an average content of 48.5cl.
Test the hypothesis that the average content per bottle is 50cl
at the 5% significance level.

Assume that the population standard deviation $\sigma = 5cl$.

Exercise 4: The impact of sample size

A machine is set up such that the average content of juice per bottle equals μ.
A sample of 100 bottles yields an average content of 48.8cl.
Test the hypothesis that the average content per bottle is 50cl
at the 5% significance level.

Compare the conclusion to that based on the 36 bottles sample.

Assume that the population standard deviation σ = 5cl

Exercise 5: One-tailed tests

A machine is set up such that the average content of juice per bottle equals μ.
A sample of 36 bottles yields an average content of 48.5cl.
Can you reject the hypothesis that the average content per bottle is less than or equal to 45cl in favour of the alternative that it exceeds 45cl (5% significance level)?

Assume that the population standard deviation σ = 5cl.

Exercise 6: Formulating H0

The manager claims that the average content of juice per bottle is less than 50cl. The machine operator disagrees. A sample of 100 bottles yields an average content of 49cl per bottle.

Does this sample allow the manager to claim he is right (5% significance level)?

Assume that the population standard deviation σ = 5 cl.

Exercise 7: CI for proportions

Sample of 80 customers
60 reply they are satisfied with the service they received

Calculate a 95% confidence interval for the proportion of satisfied customers

Observed value p (from the sample):
Standard deviation of p:
Distribution of p:
95% confidence interval for the true proportion p

Exercise 8: Confidence level of interval

The latest poll (1,100 respondents) reveals that 54% of the population supports the government's budgetary decisions. The margin of error is ± 3%.
==> Point estimate: 54%
Margin of error: ± 3%
==> Confidence interval: [51%, 57%]

Observed value p (from the sample):
Standard deviation of p:
Confidence level of interval:

Solution 1

$\bar{x}$ = 48
n=100
σ = 5
==> SD($\bar{x}$) = σ/ √n = 5/ √100 = 5 / 10 = 0.50

==> 90% confidence interval: [$\bar{x}$ - 1.64 SD($\bar{x}$), $\bar{x}$ + 1.64 SD($\bar{x}$)]
= [48 - 0.82, 48 + 0.82] = [47.18, 48.82]

95% confidence interval: [$\bar{x}$ - 1.96 SD($\bar{x}$), $\bar{x}$ + 1.96 SD($\bar{x}$)]
= [48 - 0.98, 48 + 0.98] = [47.02, 48.98]

Note: 95% confidence interval: It is common practice to round 1.96 up to 2
[$\bar{x}$ - 2 SD($\bar{x}$), $\bar{x}$ + 2 SD($\bar{x}$)]
= [48 - 1, 48 + 1] = [47, 49]

Solution 2

SD($\bar{x}$) = σ/ √n = 5/ √n
2SD($\bar{x}$) = 0.5
==> 2*5/ √n = 0.5
==> 10 = 0.5√n
==> 20 = √n
==> n = 400

Solution 3

SD($\bar{x}$) = 5/√36 = .83 ==> 2SD($\bar{x}$) = 1.66
==> Acceptance region = [μ-2SD($\bar{x}$), μ+2sd($\bar{x}$)]
=[48.34, 51.66]
and… $\bar{x}$ = 48.5

==> cannot reject hypothesis μ=50

Solution 4

SD($\bar{x}$) = 5/√100 = .50 ==> 2SD($\bar{x}$) = 1
==> Acceptance region = [μ-2SD($\bar{x}$),μ+2sd($\bar{x}$)] = [49, 51]
xb = 48.8
==> Reject hypothesis μ=5
(if μ=50 is true, we have 95% ch to have $\bar{x}$ between 49 and 51…)

Or,
(50-48.8)/0.5 1.2/0.5=2.4 > 2
==> Reject H0

// Previous case

Here: reject μ=50 on basis of observing 48.8
Before: not reject μ=50 despite observing 48.5

Solution 5

SD($\overline{x}$) = 5/√36 = .83
One-tailed test, 5% significance level ==> 5% in right tail
==> 1.64 standard deviations

==> Acceptance region = [-∞, μ+1.64sd(xb)] = [-∞, 45+1.64*0.83] = [-∞, 46.36]

$\overline{x}$ = 48.5
==> Reject hypothesis H0: $\mu \leq 45$ in favour of H1: $\mu > 45$

Note: z-score: (48.5 – 45) / 0.83 = 4.2 > 1.64 ==> Reject

Solution 6

Objective of manager: show that $\mu < 50$
==> Need to reject : $\mu \geq 50$ (remember:: Cannot 'prove' a hypothesis, only 'disprove' i.e. reject)

==> H0: $\mu \geq 50$

H0: $\mu \geq 50$
H1: $\mu < 50$
==> 1-tailed test!
==> 5% in 1 tail ==> 1.64SD($\overline{x}$)

SD(xb) = 5/√100 = .50 ==> 1.64SD($\overline{x}$) = 0.82
==> Acceptance region = [m-1.64SD($\overline{x}$),∞] = [49.18, ∞]
xb = 49
==> Reject hypothesis $\mu \geq 50$

Or,
(49-50)/0.5= -2 < -1.64
==> Reject H0

Solution 7

Observed value $\hat{p}$: 60/80 = 0.75 = 75%

Standard deviation of $\hat{p}$:
p unknown ==> use p^ instead
Proportion of satisfied customers: Var(X/n) = Var(X)/n² = $\hat{p}$ (1- $\hat{p}$)/n = 0.0023
==> N($\hat{p}$, √($\hat{p}$ (1- $\hat{p}$)/n) = N(0.75, 0.048)

Distribution of $\hat{p}$

Satisfied or not with probability p, independent opinions ==> Binomial
Approximation of Binomial by Normal distribution: required conditions
n $\hat{p}$ = 60 > 5, n(1- $\hat{p}$) = 20 > 5, 0.1< $\hat{p}$ = 0.75 < 0.9 ==> OK

95% Confidence interval for the true proportion p:
[$\hat{p}$ - 2 √ $\hat{p}$ (1- $\hat{p}$)/n, $\hat{p}$ + 2 √ $\hat{p}$ (1- $\hat{p}$)/n]
= [0.75 – 2*0.048, 0.75 + 2*0.048] = [0.654, 0.846]

Solution 8

Observed value $\hat{p}$: 0.54

Standard deviation of $\hat{p}$: √ $\hat{p}$ (1- $\hat{p}$)/n = .54*.46/1,100 = 0.015

==> "error margin of ± 3%" corresponds to ≈ 2SD($\hat{p}$)
=> 95% confidence level (3%/1.5% = 2… and The critical value 2 corresponds to a 95% CI)