

Research Article

Existence and Global Asymptotic Behavior of Singular Positive Solutions for Radial Laplacian

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The aim of this paper is to establish existence and uniqueness of a positive continuous solution to the following singular nonlinear problem. $\{-t^{1-n}(t^{n-1}u')' = a(t)u^\sigma, t \in (0, 1), \lim_{t \rightarrow 0} t^{n-1}u'(t) = 0, u(1) = 0\}$, where $n \geq 3, \sigma < 1$, and a denotes a nonnegative continuous function that might have the property of being singular at $t = 0$ and/or $t = 1$ and which satisfies certain condition associated to Karamata class. We emphasize that the nonlinearity might also be singular at $u = 0$, while the solution could blow-up at 0. Our method is based on the global estimates of potential functions and the Schauder fixed point theorem.

1. Introduction and Main Result

Nonlinear problems of the form

$$-\frac{1}{A} (Au')' = f(x, u), \quad x \in (0, 1), \quad (1)$$

$$u > 0, \quad \text{in } (0, 1),$$

where A is a positive, differentiable function on $(0, 1)$ and satisfying several suitable conditions have been studied by many researchers (see for instance [1–10]). Note that many problems in the boundary layer theory and the theory of pseudoplastic fluids can be modeled by equations of the form (1) (see for example [11, 12]).

Equations of the form (1) with $A(t) = t^{n-1}$ ($n \geq 3$), appears in a natural manner in those cases when the researcher is looking for radial solutions of Laplace operator.

For a multiple results of existence, uniqueness, and asymptotic behavior associated with similar problems, we refer the reader to [13–30] and their bibliographies.

Let us first introducing the following functional class \mathcal{K} called Karamata class.

Definition 1. Let $\eta > 1$ and L be a function defined on $(0, \eta]$. Then L belongs to the class \mathcal{K} if

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right), \quad (2)$$

where $c > 0$ and $z \in C([0, \eta])$ with $z(0) = 0$.

Here, it is pertinent to note that the functions in the class \mathcal{K} are slowly varying, and Karamata developed in [31] the initial theory in this field.

Cirstea and Rădulescu have exploited in [32] the Karamata theory to study the asymptotic and qualitative behavior near the boundary of solutions of nonlinear elliptic problems.

The aim of this paper is to address the existence, uniqueness and qualitative behavior of positive continuous solution to the following singular nonlinear problem.

$$-t^{1-n} (t^{n-1}u')' = a(t)u^\sigma, \quad t \in (0, 1), \quad (3)$$

$$\lim_{t \rightarrow 0} t^{n-1}u'(t) = 0, \quad u(1) = 0,$$

where $n \geq 3, \sigma < 1$ and a denotes a nonnegative continuous function on $(0, 1)$ that might have the property of being singular at $t = 0$ and/or $t = 1$ and which might satisfies

certain condition associated with the Karamata class \mathcal{K} . In this situation, the nonlinearity might also have the property of being singular at $u = 0$. Here we emphasize that the obtained solution may also blow-up at 0, which is not given in the previous works. Our approach relies on Karamata theory and the Schauder fixed point theorem.

Notations. (i) $\mathcal{B}((0, 1))$ (resp. $\mathcal{B}^+((0, 1))$), denotes the set of Borel (resp. nonnegative Borel) measurable functions in $(0, 1)$.

(ii) $L^1((0, 1)) := \{q \in \mathcal{B}((0, 1)), \int_0^1 |q(r)|dr < \infty\}$.

(iii) $C(X)$ (resp. $C^+(X)$), is the set of continuous (resp. nonnegative continuous) functions in a metric space X .

(iii) $C_0([0, 1]) = \{f \in C((0, 1)) : \lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 1} f(t) = 0\}$.

(iii) For $n \geq 3$,

$$C_{n-2}([0, 1]) = \{f \in C((0, 1]) : t \rightarrow t^{n-2} f(t) \in C([0, 1])\}. \quad (4)$$

(iv) For $f, g \in \mathcal{B}^+((0, 1))$, we say that $f(t) \approx g(t)$, $t \in (0, 1)$, if there exists $c > 0$ such that $(1/c)f(t) \leq g(t) \leq cf(t)$, for all $t \in (0, 1)$.

In what follows, we let $n \geq 3$, $\sigma < 1$ and assume that

(H) $a \in C^+((0, 1))$ with

$$a(t) \approx t^{-\mu} L_1(t) (1-t)^{-\lambda} L_2(1-t), \quad t \in (0, 1), \quad (5)$$

where $\mu \leq n + (2-n)\sigma$, $\lambda \leq 2$ and $L_1, L_2 \in \mathcal{K}$ defined on $(0, \eta]$ ($\eta > 1$) such that

$$\int_0^\eta t^{n+(2-n)\sigma-\mu-1} L_1(t) dt < \infty, \quad (6)$$

$$\int_0^\eta t^{1-\lambda} L_2(t) dt < \infty.$$

We introduce the function θ defined on $(0, 1]$ by

$$\theta(t) := t^{\min(0, (2-\mu)/(1-\sigma))} (\tilde{L}_1(t))^{1/(1-\sigma)} \cdot (1-t)^{\min(1, (2-\lambda)/(1-\sigma))} (\tilde{L}_2(1-t))^{1/(1-\sigma)}, \quad (7)$$

where

$$\tilde{L}_1(t) := \begin{cases} 1, & \text{if } \mu < 2, \\ \int_t^\eta \frac{L_1(s)}{s} ds, & \text{if } \mu = 2, \\ L_1(t), & \text{if } 2 < \mu < n + (2-n)\sigma, \\ \int_0^t \frac{L_1(s)}{s} ds, & \text{if } \mu = n + (2-n)\sigma, \end{cases} \quad (8)$$

and

$$\tilde{L}_2(t) := \begin{cases} 1, & \text{if } \lambda < 1, \\ \int_t^\eta \frac{L_2(s)}{s} ds, & \text{if } \lambda = 1, \\ L_2(t), & \text{if } 1 < \lambda < 2, \\ \int_0^t \frac{L_2(s)}{s} ds, & \text{if } \lambda = 2. \end{cases} \quad (9)$$

Our main result is the following.

Theorem 2. *Let $\sigma < 1$ and assume that a satisfies (H). Then problem (3) has a unique positive solution $u \in C_{n-2}([0, 1])$ satisfying for $t \in (0, 1]$,*

$$u(t) \approx \theta(t). \quad (10)$$

Remark 3. The solution obtained in Theorem 2 blow-up at 0 for $\mu > 2$.

Example 4. Let $n \geq 3$ and assume that $\mu < n$. The unique solution of the linear problem

$$-t^{1-n} (t^{n-1} u')' = t^{-\mu}, \quad t \in (0, 1), \quad (11)$$

$$\lim_{t \rightarrow 0} t^{n-1} u'(t) = 0, \quad u(1) = 0,$$

is given by

$$u_\mu(t) = \begin{cases} \frac{1}{(n-\mu)(\mu-2)} (t^{2-\mu} - 1), & \text{if } 2 < \mu < n, \\ -\frac{1}{(n-2)} \ln t, & \text{if } \mu = 2, \\ \frac{1}{(n-\mu)(2-\mu)} (1 - t^{2-\mu}), & \text{if } \mu < 2. \end{cases} \quad (12)$$

Clearly the solution blow up at 0 for $\mu \geq 2$ and also we have

$$u_\mu(t) \approx \theta(t) = \begin{cases} t^{2-\mu} (1-t), & \text{if } 2 < \mu < n, \\ (1-t) (\ln 2 - \ln t), & \text{if } \mu = 2, \\ (1-t), & \text{if } \mu < 2. \end{cases} \quad (13)$$

This implies that the global estimates obtained in Theorem 2 are optimal.

2. Karamata Class and Global Estimates

2.1. Karamata Class. It is clear to see that for some $\eta > 1$, the class \mathcal{K} is given by

$$\mathcal{K} = \left\{ L : (0, \eta] \rightarrow (0, \infty), L \in C^1((0, \eta]) \text{ and } \lim_{t \rightarrow 0^+} \frac{tL'(t)}{L(t)} = 0 \right\}. \quad (14)$$

Standard examples of functions which are elements of the class \mathcal{K} are presented below (see [33–35])

$$L(t) = \prod_{k=1}^m \left(\log_k \left(\frac{\omega}{t} \right) \right)^{\xi_k}, \quad (15)$$

$$\text{and } L(t) = \exp \left\{ \prod_{k=1}^m \left(\log_k \left(\frac{\omega}{t} \right) \right)^{\gamma_k} \right\},$$

where $\log_k t = \log \circ \log \circ \dots \circ \log t$ (k times), $\xi_k \in \mathbb{R}$, $\gamma_k \in (0, 1)$ and ω is a sufficiently large positive real number such that L is defined and positive on $(0, \eta]$, for some $\eta > 1$.

Next, we collect several properties of the Karamata functions, which will be useful in the proof of our main result.

Lemma 5 (see [34, 35]). Let $\gamma \in \mathbb{R}$ and $L \in \mathcal{K}$ defined on $(0, \eta]$, $\eta > 1$. We have

(i) If $\gamma > -1$, then $\int_0^\eta s^\gamma L(s) ds < \infty$ and $\int_0^t s^\gamma L(s) ds \underset{t \rightarrow 0^+}{\sim} t^{1+\gamma} L(t)/(1+\gamma)$.

(ii) If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s) ds = \infty$ and $\int_t^\eta s^\gamma L(s) ds \underset{t \rightarrow 0^+}{\sim} -t^{1+\gamma} L(t)/(1+\gamma)$.

Lemma 6 (see [35, 36]). (i) Let $L \in \mathcal{K}$ and $\epsilon > 0$, then we have

$$\lim_{t \rightarrow 0^+} t^\epsilon L(t) = 0. \tag{16}$$

(ii) Let $L_1, L_2 \in \mathcal{K}$ defined on $(0, \eta]$, $\eta > 1$, and $p \in \mathbb{R}$. Then the functions

$$L_1 + L_2, L_1 L_2, \text{ and } L_1^p \text{ are in } \mathcal{K}. \tag{17}$$

(iii) Let $L \in \mathcal{K}$ defined on $(0, \eta]$, $\eta > 1$, then we have

$$\lim_{t \rightarrow 0^+} \frac{L(t)}{\int_t^\eta (L(s)/s) ds} = 0. \tag{18}$$

In particular

$$t \rightarrow \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}. \tag{19}$$

If further $\int_0^\eta (L(s)/s) ds < \infty$, then we have $\lim_{t \rightarrow 0^+} (L(t)/\int_0^t (L(s)/s) ds) = 0$.

In particular

$$t \rightarrow \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}. \tag{20}$$

2.2. Global Estimates. Let $n \geq 3$, then

$$G(t, r) := \frac{1}{n-2} \frac{r^{n-1} (1 - (\max(t, r))^{n-2})}{(\max(t, r))^{n-2}}, \tag{21}$$

is the Green's function of the operator $u \rightarrow -t^{1-n}(t^{n-1}u)'$, with boundary conditions $\lim_{t \rightarrow 0} t^{n-1}u'(t) = u(1) = 0$.

Lemma 7. (i) On $[0, 1] \times [0, 1]$, we have

$$G(t, r) \approx \frac{r^{n-1} (1 - \max(t, r))}{(\max(t, r))^{n-2}}. \tag{22}$$

(ii) There exists a constant $c > 0$ such that for all $t, r \in [0, 1]$,

$$\begin{aligned} \frac{1}{c} r^{n-1} t^{n-2} (1-t)(1-r) &\leq t^{n-2} G(t, r) \\ &\leq c r^{n-1} \min(1-t, 1-r). \end{aligned} \tag{23}$$

Proof. (i) The property follows from (21) and the fact that

$$\begin{aligned} 1 - \max(t, r) &\leq 1 - (\max(t, r))^{n-2} \\ &\leq (n-2)(1 - \max(t, r)). \end{aligned} \tag{24}$$

(ii) The inequalities follow from (22) and the fact that

$$(1-t)(1-r) \leq 1 - \max(t, r) \leq \min(1-t, 1-r). \tag{25}$$

□

For $f \in \mathcal{B}^+((0, 1))$, we define the potential of f by

$$Vf(t) := \int_0^1 G(t, r) f(r) dr, \quad t \in (0, 1). \tag{26}$$

Using Lemma 7, we deduce the following.

Corollary 8. Let $f \in \mathcal{B}^+((0, 1))$, then we have

$$\begin{aligned} Vf(t) \in C_{n-2}([0, 1]) &\iff \\ \int_0^1 r^{n-1} (1-r) f(r) dr &< \infty. \end{aligned} \tag{27}$$

Proposition 9. Let $n \geq 3$ and f be a function such that the map $r \rightarrow r^{n-1}(1-r)f(r) \in C((0, 1)) \cap L^1((0, 1))$, then Vf belongs to $C_{n-2}([0, 1])$ and it is the unique solution of the problem

$$\begin{aligned} -t^{1-n} (t^{n-1}u)' &= f(t), \quad t \in (0, 1), \\ \lim_{t \rightarrow 0} t^{n-1}u'(t) &= 0, \quad u(1) = 0. \end{aligned} \tag{28}$$

Proof. From Corollary 8, the function Vf is in $C_{n-2}([0, 1])$.

Using (21), we have for $t \in (0, 1)$,

$$\begin{aligned} Vf(t) &= \int_0^1 G(t, r) f(r) dr \\ &= \frac{1}{n-2} (t^{2-n} - 1) \int_0^t r^{n-1} f(r) dr \\ &\quad + \frac{1}{n-2} \int_t^1 r (1 - r^{n-2}) f(r) dr \\ &= J_1(t) + J_2(t). \end{aligned} \tag{29}$$

Since the function $r \rightarrow r^{n-1}(1-r)f(r) \in C((0, 1)) \cap L^1((0, 1))$, then $J_1(t)$ and $J_2(t)$ are differentiable on $(0, 1)$.

By simple calculation, we obtain

$$(Vf)'(t) = -t^{1-n} \int_0^t r^{n-1} f(r) dr. \tag{30}$$

That is

$$t^{n-1} (Vf)'(t) = - \int_0^t r^{n-1} f(r) dr. \tag{31}$$

By differentiating (31), we obtain for $t \in (0, 1)$,

$$-t^{1-n} (t^{n-1} (Vf)')'(t) = f(t). \tag{32}$$

Using (31) and the fact that $Vf \in C_{n-2}([0, 1])$, we obtain

$$\lim_{t \rightarrow 0} t^{n-1} (Vf)'(t) = (Vf)'(1) = 0. \tag{33}$$

Finally, we prove the uniqueness. Let $u, v \in C_{n-2}([0, 1])$ be two solutions of (28) and put $w = u - v$. Then $w \in C_{n-2}([0, 1])$ and $(t^{n-1}w')' = 0$. Since $\lim_{t \rightarrow 0} t^{n-1}w'(t) = 0$, we deduce that $t^{n-1}w'(t) = 0$ and therefore $w(t) = c$. Using the fact $w(1) = 0$, we deduce that $w = 0$. That is $u = v$. \square

Proposition 10. Let $\gamma \leq n$, $\nu \leq 2$ and $L_3, L_4 \in \mathcal{K}$ such that

$$\int_0^n r^{n-\gamma-1} L_3(r) dr < \infty \quad (34)$$

and $\int_0^n r^{1-\nu} L_4(r) dr < \infty$.

Put

$$b(t) = t^{-\gamma} L_3(t) (1-t)^{-\nu} L_4(1-t), \quad \text{for } t \in (0, 1). \quad (35)$$

Then for $t \in (0, 1]$,

$$Vb(t) \approx t^{\min(0, 2-\gamma)} \tilde{L}_3(t) (1-t)^{\min(1, 2-\nu)} \tilde{L}_4(1-t), \quad (36)$$

where

$$\tilde{L}_3(t) := \begin{cases} 1, & \text{if } \gamma < 2, \\ \int_t^n \frac{L_3(r)}{r} dr, & \text{if } \gamma = 2, \\ L_3(t), & \text{if } 2 < \gamma < n, \\ \int_0^t \frac{L_3(r)}{r} dr, & \text{if } \gamma = n, \end{cases} \quad (37)$$

and

$$\tilde{L}_4(t) := \begin{cases} 1, & \text{if } \nu < 1, \\ \int_t^n \frac{L_4(r)}{r} dr, & \text{if } \nu = 1, \\ L_4(t), & \text{if } 1 < \nu < 2, \\ \int_0^t \frac{L_4(r)}{r} dr, & \text{if } \nu = 2. \end{cases} \quad (38)$$

Proof. For $t \in (0, 1]$, we have

$$Vb(t) = \int_0^1 G(t, r) b(r) dr. \quad (39)$$

Using (22), we obtain that

$$\begin{aligned} Vb(t) &\approx (1-t)t^{2-n} \int_0^t r^{n-\gamma-1} (1-r)^{-\nu} L_3(r) L_4(1-r) dr \\ &\quad + \int_t^1 r^{1-\gamma} (1-r)^{1-\nu} L_3(r) L_4(1-r) dr. \end{aligned} \quad (40)$$

In what follows, we distinguish two cases.

Case 1 ($0 < t \leq 1/2$). In this case $1-t \approx 1$. So we obtain

$$\begin{aligned} Vb(t) &\approx t^{2-n} \int_0^t r^{n-\gamma-1} L_3(r) dr \\ &\quad + \left(\int_t^{1/2} r^{1-\gamma} (1-r)^{1-\nu} L_3(r) L_4(1-r) dr \right. \\ &\quad \left. + \int_{1/2}^1 r^{1-\gamma} (1-r)^{1-\nu} L_3(r) L_4(1-r) dr \right) \\ &\approx t^{2-n} \int_0^t r^{n-\gamma-1} L_3(r) dr + \left(\int_t^{1/2} r^{1-\gamma} L_3(r) dr \right. \\ &\quad \left. + \int_{1/2}^1 (1-r)^{1-\nu} L_4(1-r) dr \right) \\ &= t^{2-n} \int_0^t r^{n-\gamma-1} L_3(r) dr + \left(\int_t^{1/2} r^{1-\gamma} L_3(r) dr \right. \\ &\quad \left. + \int_0^{1/2} r^{1-\nu} L_4(r) dr \right). \end{aligned} \quad (41)$$

Since $\int_0^n r^{1-\nu} L_4(r) dr < \infty$, we deduce that

$$\begin{aligned} Vb(t) &\approx t^{2-n} \int_0^t r^{n-\gamma-1} L_3(r) dr \\ &\quad + \left(1 + \int_t^{1/2} r^{1-\gamma} L_3(r) dr \right). \end{aligned} \quad (42)$$

Using Lemma 5 and (34), we deduce that

$$\int_0^t r^{n-\gamma-1} L_3(r) dr \approx \begin{cases} t^{n-\gamma} L_3(t), & \text{if } \gamma < n, \\ \int_0^t \frac{L_3(r)}{r} dr, & \text{if } \gamma = n \end{cases} \quad (43)$$

and

$$\begin{aligned} 1 + \int_t^{1/2} r^{1-\gamma} L_3(r) dr &\approx \begin{cases} 1, & \text{if } \gamma < 2, \\ \int_t^\eta \frac{L_3(r)}{r} dr, & \text{if } \gamma = 2, \\ t^{2-\gamma} L_3(t), & \text{if } 2 < \gamma \leq n. \end{cases} \end{aligned} \quad (44)$$

Hence, it follows by Lemmas 5, 6 and (34) that, for $0 < t \leq 1/2$,

$$Vb(t) \approx \begin{cases} 1, & \text{if } \gamma < 2, \\ \int_t^\eta \frac{L_3(r)}{r} dr, & \text{if } \gamma = 2, \\ t^{2-\gamma} L_3(t), & \text{if } 2 < \gamma < n, \\ t^{2-\gamma} \int_0^t \frac{L_3(r)}{r} dr, & \text{if } \gamma = n. \end{cases} \quad (45)$$

That is

$$Vb(t) \approx t^{\min(0, 2-\gamma)} \tilde{L}_3(t). \quad (46)$$

Case 2 ($1/2 < t \leq 1$). In this case, $t \approx 1$. Therefore, we have

$$\begin{aligned}
 Vb(t) &\approx t^{2-n} \left(\int_0^{1/2} r^{n-\gamma-1} (1-t)(1-r)^{-\gamma} L_3(r) \right. \\
 &\quad \cdot L_4(1-r) dr + \int_{1/2}^t r^{n-\gamma-1} (1-t)(1-r)^{-\gamma} \\
 &\quad \cdot L_3(r) L_4(1-r) dr \Big) + \int_t^1 r^{1-\gamma} (1-r)^{1-\gamma} L_3(r) \\
 &\quad \cdot L_4(1-r) dr \approx (1-t) \left(\int_0^{1/2} r^{n-\gamma-1} L_3(r) dr \right. \\
 &\quad + \int_{1/2}^t (1-r)^{-\gamma} L_4(1-r) dr \Big) + \int_t^1 (1-r)^{1-\gamma} \\
 &\quad \cdot L_4(1-r) dr = (1-t) \left(\int_0^{1/2} r^{n-\gamma-1} L_3(r) dr \right. \\
 &\quad + \int_{1-t}^{1/2} r^{-\gamma} L_4(r) dr \Big) + \int_0^{1-t} r^{1-\gamma} L_4(r) dr.
 \end{aligned} \tag{47}$$

Since $\int_0^\eta r^{n-\gamma-1} L_3(r) dr < \infty$, we deduce that

$$\begin{aligned}
 Vb(t) &\approx (1-t) \left(1 + \int_{1-t}^{1/2} r^{-\gamma} L_4(r) dr \right) \\
 &\quad + \int_0^{1-t} r^{1-\gamma} L_4(r) dr.
 \end{aligned} \tag{48}$$

Using again Lemma 5 and hypothesis (34), we deduce that

$$\begin{aligned}
 &\int_0^{1-t} r^{1-\gamma} L_4(r) dr \\
 &\approx \begin{cases} (1-t)^{2-\gamma} L_4(1-t), & \text{if } \gamma < 2, \\ \int_0^{1-t} \frac{L_4(r)}{r} dr, & \text{if } \gamma = 2 \end{cases} \tag{49}
 \end{aligned}$$

and

$$\begin{aligned}
 &1 + \int_{1-t}^{1/2} r^{-\gamma} L_4(r) dr \\
 &\approx \begin{cases} 1, & \text{if } \gamma < 1, \\ \int_{1-t}^\eta \frac{L_4(r)}{r} dr, & \text{if } \gamma = 2, \\ (1-t)^{1-\gamma} L_4(1-t), & \text{if } 1 < \gamma \leq 2. \end{cases} \tag{50}
 \end{aligned}$$

Hence, it follows by Lemmas 5, 6 and hypothesis (34) that, for $1/2 < t \leq 1$, we get

$$Vb(t) \approx \begin{cases} (1-t), & \text{if } \gamma < 1, \\ (1-t) \int_{1-t}^\eta \frac{L_4(r)}{r} dr, & \text{if } \gamma = 1, \\ (1-t)^{2-\gamma} L_4(1-t), & \text{if } 1 < \gamma < 2, \\ \int_0^{1-t} \frac{L_4(r)}{r} dr, & \text{if } \gamma = 2. \end{cases} \tag{51}$$

That is

$$Vb(t) \approx (1-t)^{\min(1, 2-\gamma)} \tilde{L}_4(1-t). \tag{52}$$

Combining (46) and (52), we obtain for $t \in (0, 1]$,

$$Vb(t) \approx t^{\min(0, 2-\gamma)} \tilde{L}_3(t) (1-t)^{\min(1, 2-\gamma)} \tilde{L}_4(1-t). \tag{53}$$

This ends the proof. \square

Proposition 11. Assume that condition (H) is satisfied. Then for $t \in (0, 1]$, we have

$$Vp(t) \approx \theta(t), \tag{54}$$

where $p(r) := a(r)\theta^\sigma(r)$.

Proof. Let a be a function satisfying (H). Using (5) and (7), we obtain

$$\begin{aligned}
 p(t) &\approx t^{-\gamma} L_1(t) (\tilde{L}_1(t))^{\sigma/(1-\sigma)} (1-t)^{-\nu} L_2(1-t) \\
 &\quad \cdot (\tilde{L}_2(1-t))^{\sigma/(1-\sigma)},
 \end{aligned} \tag{55}$$

where $\gamma = \mu - \min(0, (2-\mu)/(1-\sigma))\sigma$ and $\nu = \lambda - \min(1, (2-\lambda)/(1-\sigma))\sigma$.

Since $\mu \leq n + (2-n)\sigma$ and $\lambda \leq 2$, then one can easy check that $\gamma \leq n$ and $\nu \leq 2$.

Now using Lemmas 5, 6 and Proposition 10 with $L_3 = L_1(\tilde{L}_1)^{\sigma/(1-\sigma)} \in \mathcal{H}$ and $L_4 = L_2(\tilde{L}_2)^{\sigma/(1-\sigma)} \in \mathcal{H}$, we deduce that for each $t \in (0, 1]$,

$$Vp(t) \approx t^{\min(0, 2-\gamma)} \tilde{L}_3(t) (1-t)^{\min(1, 2-\nu)} \tilde{L}_4(1-t). \tag{56}$$

Since $\min(0, 2-\gamma) = \min(0, (2-\mu)/(1-\sigma))$ and $\min(1, 2-\nu) = \min(1, (2-\lambda)/(1-\sigma))$, then we deduce

$$\begin{aligned}
 Vp(t) &\approx t^{\min(0, (2-\mu)/(1-\sigma))} \tilde{L}_3(t) (1-t)^{\min(1, (2-\lambda)/(1-\sigma))} \\
 &\quad \cdot \tilde{L}_4(1-t) \approx \theta(t).
 \end{aligned} \tag{57}$$

This completes the proof. \square

3. Existence Results

3.1. Preliminary Results. For $\alpha \geq 0$, we denote by (P_α) the following problem

$$(P_\alpha) \begin{cases} -t^{1-n} (t^{n-1} u')' = a(t) u^\sigma, & \text{in } (0, 1), \\ \lim_{t \rightarrow 0} t^{n-1} u'(t) = (2-n)\alpha, \\ u(1) = \alpha. \end{cases} \tag{58}$$

Next, we establish several facts to be used within the proof of the main result.

Lemma 12. Let $\sigma < 0$, $0 \leq \alpha \leq \beta$ and $u_\alpha, u_\beta \in C((0, 1]) \cap C^1((0, 1))$ be two positive functions satisfying

$$\begin{aligned}
 &-t^{1-n} (t^{n-1} u'_\alpha)' \leq a(t) u_\alpha^\sigma, \quad \text{in } (0, 1), \\
 &\lim_{t \rightarrow 0} t^{n-1} u'_\alpha(t) = (2-n)\alpha, \\
 &u_\alpha(1) = \alpha.
 \end{aligned} \tag{59}$$

and

$$\begin{aligned} -t^{1-n} \left(t^{n-1} u'_\beta \right)' &\geq a(t) u_\beta^\sigma, \quad \text{in } (0, 1), \\ \lim_{t \rightarrow 0} t^{n-1} u'_\beta(t) &= (2-n) \beta, \\ u_\beta(1) &= \beta. \end{aligned} \quad (60)$$

Then $u_\alpha \leq u_\beta$ in $(0, 1]$.

Proof. Let $u(t) := u_\alpha(t) - u_\beta(t)$ for $t \in (0, 1)$. Assume that $u(t_0) > 0$ for some $t_0 \in (0, 1)$. Then there exists an interval $(t_1, t_2) \subset [0, 1]$ containing t_0 such that $u(t) > 0$, on (t_1, t_2) with $u(t_2) = 0$ and $u(t_1) = 0$ or $t_1 = 0$.

On the other hand, since $\sigma < 0$, then we have $u_\beta^\sigma(t) > u_\alpha^\sigma(t)$ for $t \in (t_1, t_2)$. So

$$\begin{aligned} t^{1-n} \left(t^{n-1} u' \right)' &= t^{1-n} \left(t^{n-1} u'_\alpha \right)' - t^{1-n} \left(t^{n-1} u'_\beta \right)' \\ &\geq a(t) \left(u_\beta^\sigma - u_\alpha^\sigma \right) > 0 \quad \text{on } (t_1, t_2). \end{aligned} \quad (61)$$

So the function $t \rightarrow t^{n-1} u'$ is nondecreasing on (t_1, t_2) with $\lim_{t \rightarrow t_1} t^{n-1} u'(t) \geq 0$.

Hence, the function u is nondecreasing on (t_1, t_2) with $u(t_0) > 0$ and $u(t_2) = 0$. This gives a contradiction. Therefore, $u_\alpha \leq u_\beta$. \square

Proposition 13. *Let $\sigma < 0$, and assume that hypothesis (H) is satisfied. Then for each $\alpha > 0$, problem (P_α) has a unique positive solution $u_\alpha \in C((0, 1]) \cap C^1((0, 1))$ satisfying for $t \in (0, 1)$*

$$u_\alpha(t) = \alpha t^{2-n} + \int_0^1 G(t, r) a(r) u_\alpha^\sigma(r) dr. \quad (62)$$

In particular, $\lim_{t \rightarrow 0} t^{n-2} u_\alpha(t) = \alpha$.

Proof. Let $\sigma < 0$, $\alpha > 0$ and assume that the function a satisfies hypothesis (H). Note that the function

$$\begin{aligned} t &\longrightarrow h(t) \\ &:= t^{n-2} \int_0^1 G(t, r) a(r) r^{(2-n)\sigma} dr \quad \text{is in } C_0([0, 1]). \end{aligned} \quad (63)$$

Indeed, since for each $r > 0$, the function $t \rightarrow t^{n-2} G(t, r) \in C_0([0, 1])$, it follows from (23), (6) and the convergence dominated theorem that $h \in C_0([0, 1])$.

Let $\beta := \alpha + \alpha^\sigma \|h\|_\infty$ and Λ be the closed convex set given by

$$\Lambda = \{v \in C([0, 1]) : \alpha \leq v \leq \beta\}. \quad (64)$$

Define the operator T on Λ by

$$Tv(t) = \alpha + t^{n-2} \int_0^1 G(t, r) a(r) r^{(2-n)\sigma} v^\sigma(r) dr. \quad (65)$$

Since $\sigma < 0$, then for all $v \in \Lambda$, we have

$$\alpha \leq Tv \leq \beta. \quad (66)$$

On the other hand, we have for all $t_1, t_2 \in [0, 1]$

$$\begin{aligned} &|Tv(t_1) - Tv(t_2)| \\ &\leq \alpha^\sigma \int_0^1 \left| t_1^{n-2} G(t_1, r) - t_2^{n-2} G(t_2, r) \right| a(r) r^{(2-n)\sigma} dr. \end{aligned} \quad (67)$$

Since, for each $r > 0$, the function $t \rightarrow t^{n-2} G(t, r) \in C_0([0, 1])$, then we deduce by (23), (6) and the convergence dominated theorem that $T(\Lambda)$ is equicontinuous in $[0, 1]$. In particular, for all $v \in \Lambda$, $T(\Lambda) \subset C([0, 1])$ and so $T(\Lambda) \subset \Lambda$. Moreover, since the family $\{Tv(t), v \in \Lambda\}$ is uniformly bounded in $[0, 1]$, then by Ascoli's theorem that $T(\Lambda)$ becomes relatively compact in $C([0, 1])$. Next, we prove the continuity of T in Λ . Let $(v_k)_k \subset \Lambda$ and $v \in \Lambda$ such that $\|v_k - v\|_\infty \rightarrow 0$ as $k \rightarrow \infty$. Then we have

$$\begin{aligned} &|Tv_k(t) - Tv(t)| \\ &\leq \int_0^1 t^{n-2} G(t, r) a(r) r^{(2-n)\sigma} |v_k^\sigma(r) - v^\sigma(r)| dr. \end{aligned} \quad (68)$$

Now, since

$$|v_k^\sigma(r) - v^\sigma(r)| \leq 2\alpha^\sigma, \quad (69)$$

we deduce by convergence dominated theorem that

$$\forall t \in [0, 1], \quad Tv_k(t) \rightarrow Tv(t) \quad \text{as } k \rightarrow \infty. \quad (70)$$

Since $T(\Lambda)$ is relatively compact in $C([0, 1])$, we obtain

$$\|Tv_k - Tv\|_\infty \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (71)$$

So T is a compact mapping Λ to itself. Therefore, by the Schauder fixed point theorem, there exists $v_\alpha \in \Lambda$ such that for each $t \in [0, 1]$

$$v_\alpha(t) = \alpha + t^{n-2} \int_0^1 G(t, r) a(r) r^{(2-n)\sigma} v_\alpha^\sigma(r) dr. \quad (72)$$

Put $u_\alpha(t) = t^{2-n} v_\alpha$, for $t \in (0, 1]$. Then $u_\alpha \in C((0, 1])$ and we have

$$u_\alpha(t) = \alpha t^{2-n} + \int_0^1 G(t, r) a(r) u_\alpha^\sigma(r) dr, \quad (73)$$

and

$$\alpha t^{2-n} \leq u_\alpha(t) \leq \beta t^{2-n}. \quad (74)$$

We have for all $r \in (0, 1)$,

$$|a(r) u_\alpha^\sigma(r)| \leq \alpha^\sigma a(r) r^{(2-n)\sigma}. \quad (75)$$

Now since by hypothesis (H) the function $r \rightarrow r^{n-1} (1-r) a(r) r^{(2-n)\sigma} \in C((0, 1)) \cap L^1((0, 1))$, we deduce from (73) and Proposition 9 that u_α is a solution of problem (P_α) . By Lemma 12, we obtain the uniqueness.

Finally, using (73) and the fact that

$$0 \leq t^{n-2} \int_0^1 G(t, r) a(r) u_\alpha^\sigma(r) dr \leq \alpha^\sigma h(t) \quad (76)$$

$$\in C_0([0, 1]),$$

we deduce that $\lim_{t \rightarrow 0} t^{n-2} u_\alpha(t) = \alpha$. \square

From Lemma 12 and (73) we deduce the following

Corollary 14. *Let $\sigma < 0$, and assume that hypothesis (H) is satisfied. For $0 < \alpha_1 \leq \alpha_2$, we denote by $u_{\alpha_i} \in C((0, 1]) \cap C^1((0, 1))$ the unique positive solution of the problem (P_{α_i}) . Then we have*

$$0 \leq u_{\alpha_2}(t) - u_{\alpha_1}(t) \leq (\alpha_2 - \alpha_1)t^{2-n}, \quad \text{for } t \in (0, 1]. \quad (77)$$

Theorem 15. *Let $\sigma < 0$. Under hypothesis (H), problem (3) has a unique positive solution $u \in C((0, 1]) \cap C^1((0, 1))$ satisfying for $t \in (0, 1)$*

$$u(t) = \int_0^1 G(t, r) a(r) u^\sigma(r) dr. \quad (78)$$

Proof. Let $(\alpha_k)_k \subset (0, \infty)$ be a sequence that decreases to zero. Let u_k be the unique positive continuous solution of the problem (P_{α_k}) . By Lemma 12 (or Corollary 14), the sequence $(u_k)_k$ decreases to a function u , and from (73) the sequence $(u_k - \alpha_k t^{2-n})_k$ increases to u . Using this fact and (73), we obtain for each $t \in (0, 1)$,

$$\begin{aligned} u(t) &\geq u_k(t) - \alpha_k t^{2-n} = \int_0^1 G(t, r) a(r) u_k^\sigma(r) dr \\ &\geq \beta_k^\sigma \int_0^1 G(t, r) a(r) r^{(2-n)\sigma} dr > 0, \end{aligned} \quad (79)$$

where $\beta_k := \alpha_k + \alpha_k^\sigma \|h\|_\infty$ and h is given by (63).

By the monotone convergence theorem, we obtain

$$u(t) = \int_0^1 G(t, r) a(r) u^\sigma(r) dr. \quad (80)$$

Since $u = \inf_k u_k = \sup_k (u_k - \alpha_k t^{2-n})$, then u is upper and lower semi-continuous function on $(0, 1]$ and so $u \in C((0, 1])$.

We claim that u is a solution of problem (3).

From (23), there exists $c > 0$, such that for all $t \in [0, 1]$,

$$\begin{aligned} \frac{1}{c} (1-t) \int_0^1 r^{n-1} (1-r) a(r) u^\sigma(r) dr \\ \leq \int_0^1 G(t, r) a(r) u^\sigma(r) dr = u(t). \end{aligned} \quad (81)$$

In particular,

$$\int_0^1 r^{n-1} (1-r) a(r) u^\sigma(r) dr \leq 2cu \left(\frac{1}{2}\right) < \infty. \quad (82)$$

So the function $r \rightarrow r^{n-1}(1-r)a(r)u^\sigma(r) \in C((0, 1]) \cap L^1((0, 1))$. By Proposition 9, u is a solution of problem (3). Finally, by Lemma 12, we obtain the uniqueness. \square

3.2. Proof of Theorem 2. Assume that hypothesis (H) is fulfilled. Let θ be the function defined in (7). By Proposition 11, there exists $M \geq 1$ such that for each $t \in (0, 1)$,

$$\frac{1}{M} \theta(t) \leq Vp(t) \leq M\theta(t), \quad (83)$$

where $p(r) := a(r)\theta^\sigma(r)$. We will break up the proof in two cases.

Case 1 ($\sigma < 0$). Let u be the solution of problem (3) given by Theorem 15. We claim that u satisfies (10).

Let $c = M^{-\sigma/(1-\sigma)}$ and put $v := (1/c)Vp$.

Using hypothesis (H) and Lemmas 5, 6, we deduce that

$$\int_0^1 r^{n-1} (1-r) p(r) dr < \infty. \quad (84)$$

So, it follows by Proposition 10 and (83), that v satisfies

$$\begin{aligned} -t^{1-n} (t^{n-1} v')' &\leq v(t), \quad t \in (0, 1), \\ \lim_{t \rightarrow 0} t^{n-1} v'(t) &= 0, \quad v(1) = 0. \end{aligned} \quad (85)$$

Therefore by Lemma 12, we have

$$\frac{1}{c} Vp \leq u. \quad (86)$$

Similarly, we show that

$$u \leq cVp. \quad (87)$$

Now since by (83), $Vp(t) \approx \theta(t)$, we deduce that $u(t) \approx \theta(t)$.

Case 2 ($0 \leq \sigma < 1$). We shall use a fixed point argument to construct a solution of problem (3). Let $\varphi(t) = t^{n-2}\theta(t)$, for $t \in [0, 1]$. By (83), we have

$$\frac{1}{M} \varphi(t) \leq t^{n-2} Vp(t) \leq M\varphi(t). \quad (88)$$

Put $c = M^{1/(1-\sigma)}$ and consider the closed convex set given by

$$E = \left\{ v \in C([0, 1]), \frac{1}{c} \varphi \leq v \leq c\varphi \right\}. \quad (89)$$

Clearly $\varphi \in E$.

We define the operator \mathcal{F} on E by

$$\begin{aligned} \mathcal{F}v(t) &:= t^{n-2} \int_0^1 G(t, r) a(r) r^{(2-n)\sigma} v^\sigma(r) dr, \\ &t \in [0, 1]. \end{aligned} \quad (90)$$

Using (88), for $v \in E$, we have

$$\frac{1}{c} \varphi \leq \mathcal{F}v \leq c\varphi. \quad (91)$$

Since for each $r > 0$, the function $t \rightarrow t^{n-2}G(t, r)$ is in $C_0([0, 1])$, it follows by (23), (84) and the convergence dominated theorem that $\mathcal{F}(E) \subset E$.

Let $(v_k)_k \subset C([0, 1])$ defined by

$$v_0 = \frac{1}{c} \varphi \quad (92)$$

and $v_{k+1} = \mathcal{F}v_k$, for $k \in \mathbb{N}$.

Since the operator \mathcal{F} is nondecreasing and $\mathcal{F}(E) \subset E$, we deduce that

$$v_0 \leq v_1 \leq v_2 \leq \dots \leq v_k \leq v_{k+1} \leq c\varphi. \tag{93}$$

Therefore, the sequence $(v_k)_k$ converges by the convergence monotone theorem to a function v satisfying for each $t \in [0, 1]$

$$\frac{1}{c}\varphi(t) \leq v(t) \leq c\varphi(t) \tag{94}$$

$$\text{and } v(t) = t^{n-2} \int_0^1 G(t, r) a(r) r^{(2-n)\sigma} v^\sigma(r) dr.$$

Using (23) and (84), we prove that $v \in C([0, 1])$.

Put $u(t) = t^{2-n}v(t)$. Then $u \in C_{n-2}([0, 1])$ and satisfies the equation

$$u(t) = V(au^\sigma)(t). \tag{95}$$

Now since the function $r \mapsto r^{n-1}(1-r)a(t)u^\sigma(r) \in C((0, 1]) \cap L^1((0, 1))$, it follows from Proposition 9 that u is a positive solution of problem (3) satisfying for $t \in (0, 1]$,

$$u(t) \approx \theta(t). \tag{96}$$

For the uniqueness, let u and v be two arbitrary solutions of problem (3) in the cone

$$S = \{u \in C_{n-2}([0, 1]), u(t) \approx \theta(t)\}. \tag{97}$$

There exists a constant $m \geq 1$ such that

$$\frac{1}{m} \leq \frac{u}{v} \leq m, \text{ in } (0, 1). \tag{98}$$

So the set $J := \{m \geq 1 : 1/m \leq u/v \leq m\}$ is not empty. Let $m_0 := \inf J$. Then $m_0 \geq 1$ and we have $u^\sigma \leq m_0^\sigma v^\sigma$.

Let $z := m_0^\sigma v - u$. Then

$$-t^{1-n} (t^{n-1} z')' = a(t) (m_0^\sigma v^\sigma - u^\sigma) \geq 0 \text{ in } (0, 1), \tag{99}$$

$$\lim_{t \rightarrow 0} t^{n-1} z'(t) = 0, \quad z(1) = 0,$$

Which implies by Proposition 9 that $m_0^\sigma v - u = V(a(m_0^\sigma v^\sigma - u^\sigma)) \geq 0$. By symmetry, we obtain that $m_0^\sigma u \geq v$. Hence, $m_0^\sigma \in J$. Using the fact that $\sigma < 1$, we get $m_0 = 1$. Then, we conclude that $u = v$.

This completes the proof of Theorem 2.

Example 16. Let $\sigma < 1$ and $a \in C((0, 1))$ such that

$$a(r) \approx r^{-\mu} \left(\log\left(\frac{3}{r}\right)\right)^{-\beta} (1-r)^{-2} \left(\log\left(\frac{3}{1-r}\right)\right)^{-2}, \tag{100}$$

where $\mu < n + (2 - n)\sigma$ and $\beta \in \mathbb{R}$. Then, by Theorem 2, problem (3) has a unique positive solution u in $C_{n-2}([0, 1])$ satisfying the following estimates:

(i) If $2 < \mu < n + (2 - n)\sigma$, then for $t \in (0, 1)$,

$$u(t) \approx t^{(2-\mu)/(1-\sigma)} \left(\log\left(\frac{3}{t}\right)\right)^{-\beta/(1-\sigma)} \cdot \left(\log\left(\frac{3}{1-t}\right)\right)^{-1/(1-\sigma)}. \tag{101}$$

(ii) If $\mu = 2$ and $\beta > 1$ or $\mu < 2$, then for $t \in (0, 1)$,

$$u(t) \approx \left(\log\left(\frac{3}{1-t}\right)\right)^{-1/(1-\sigma)}. \tag{102}$$

(iii) If $\mu = 2$ and $\beta = 1$, then for $t \in (0, 1)$,

$$u(t) \approx \left(\log_2\left(\frac{3}{t}\right)\right)^{1/(1-\sigma)} \left(\log\left(\frac{3}{1-t}\right)\right)^{-1/(1-\sigma)}. \tag{103}$$

(iv) If $\mu = 2$ and $\beta < 1$, then for $t \in (0, 1)$,

$$u(t) \approx \left(\log\left(\frac{3}{t}\right)\right)^{(1-\beta)/(1-\sigma)} \left(\log\left(\frac{3}{1-t}\right)\right)^{-1/(1-\sigma)}. \tag{104}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] R. P. Agarwal and D. O'Regan, "Twin solutions to singular Dirichlet problems," *Journal of Mathematical Analysis and Applications*, vol. 240, no. 2, pp. 433–445, 1999.
- [2] R. P. Agarwal and D. O'Regan, "Existence theory for single and multiple solutions to singular positive boundary value problems," *Journal of Differential Equations*, vol. 175, no. 2, pp. 393–414, 2001.
- [3] I. Bachar, S. Ben Othman, and H. Mâagli, "Radial solutions for the p-Laplacian equation," *Nonlinear Analysis*, vol. 70, no. 6, pp. 2198–2205, 2009.
- [4] H. Mâagli, "On the solutions of a singular nonlinear periodic boundary value problem," *Potential Analysis*, vol. 14, pp. 437–447, 2001.
- [5] H. Mâagli and S. Masmoudi, "Sur les solutions d'un opérateur différentiel singulier semi-linéaire," *Potential Analysis*, vol. 10, pp. 289–304, 1999.
- [6] H. Mâagli and S. Masmoudi, "Existence theorem of nonlinear singular boundary value problem," *Nonlinear Analysis*, vol. 46, pp. 465–473, 2001.
- [7] H. Mâagli and N. Zeddini, "Positive solutions for a singular nonlinear Dirichlet problem," *Nonlinear Studies*, vol. 10, pp. 295–306, 2003.

- [8] S. Masmoudi and S. Zermani, "Existence and boundary behavior of positive solutions for a Sturm-Liouville problem," *Opuscula Mathematica*, vol. 36, no. 5, pp. 613–629, 2016.
- [9] D. O'Regan, "Existence principles and theory for singular Dirichlet boundary value problems," *Differential Equations Dynamical Systems* 3, vol. 3, pp. 289–304, 1995.
- [10] S. D. Taliaferro, "A nonlinear singular boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 3, no. 6, pp. 897–904, 1979.
- [11] A. Callegari and A. Nachman, "Some singular, nonlinear differential equations arising in boundary layer theory," *Journal of Mathematical Analysis and Applications*, vol. 64, no. 1, pp. 96–105, 1978.
- [12] A. Nachman and A. Callegari, "A nonlinear singular boundary value problem in the theory of pseudoplastic fluids," *SIAM Journal on Applied Mathematics*, vol. 38, no. 2, pp. 275–281, 1980.
- [13] I. Bachar, "Combined effects in nonlinear singular second-order differential equations on the half-line," *Electronic Journal of Differential Equations*, vol. 2015, no. 235, pp. 1–12, 2015.
- [14] I. Bachar, H. Mâagli, and V. D. Radulescu, "Singular solutions of a nonlinear elliptic equation in a punctured domain," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 94, pp. 1–19, 2017.
- [15] M. Cencelj, D. Repovš, and Ž. Virk, "Multiple perturbations of a singular eigenvalue problem," *Nonlinear Analysis*, vol. 119, pp. 37–45, 2015.
- [16] F.-C. Cirstea and V. Radulescu, "Boundary blow-up in nonlinear elliptic equations of Bieberbach-Rademacher type," *Transactions of the American Mathematical Society*, vol. 359, no. 7, pp. 3275–3286, 2007.
- [17] S. Dumont, L. Dupaigne, O. Goubet, and V. Radulescu, "Back to the Keller-Osserman condition for boundary blow-up solutions," *Advanced Nonlinear Studies*, vol. 7, no. 2, pp. 271–298, 2007.
- [18] M. Ghergu and V. D. Radulescu, *PDEs Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monographs in Mathematics, Springer Verlag, Heidelberg, Germany, 2012.
- [19] M. Ghergu and V. D. Radulescu, *Singular elliptic problems: bifurcation and asymptotic analysis*, vol. 37 of *Oxford Lecture Series in Mathematics and its Applications*, The Clarendon Press, Oxford University Press, Oxford, UK, 2008.
- [20] N. Kawano, E. Yanagida, and S. Yotsutani, "Structure theorems for positive radial solutions to $\Delta u + K(x)u^p = 0$ in \mathbb{R}^n ," *Funkcialaj Ekvacioj*, vol. 36, pp. 557–579, 1993.
- [21] H. Mâagli, "Asymptotic behavior of positive solutions of a semilinear Dirichlet problem," *Nonlinear Analysis*, vol. 74, pp. 2941–2947, 2011.
- [22] H. Mâagli, S. Ben Othman, and S. Dridi, "Existence and asymptotic behavior of ground state solutions of semilinear elliptic system," *Advances in Nonlinear Analysis*, vol. 6, no. 3, pp. 301–315, 2017.
- [23] M. Nursultanov and G. Rozenblum, "Eigenvalue asymptotics for the Sturm-Liouville operator with potential having a strong local negative singularity," *Opuscula Mathematica*, vol. 37, no. 1, pp. 109–139, 2017.
- [24] V. D. Radulescu, "Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods," in *Contemporary Mathematics and Its Applications*, vol. 6, Hindawi Publishing Corporation, 2008.
- [25] D. Repovš, "Singular solutions of perturbed logistic-type equations," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4414–4422, 2011.
- [26] D. Repovš, "Asymptotics for singular solutions of quasilinear elliptic equations with an absorption term," *Journal of Mathematical Analysis and Applications*, vol. 395, no. 1, pp. 78–85, 2012.
- [27] H. Usami, "On a singular elliptic boundary value problem in a ball," *Nonlinear Analysis*, vol. 13, no. 10, pp. 1163–1170, 1989.
- [28] N. Zeddini, R. Alsaedi, and H. Mâagli, "Exact boundary behavior of the unique positive solution to some singular elliptic problems," *Nonlinear Analysis*, vol. 89, pp. 146–156, 2013.
- [29] Z. Zhang and B. Li, "The boundary behavior of the unique solution to a singular Dirichlet problem," *Journal of Mathematical Analysis and Applications*, vol. 391, no. 1, pp. 278–290, 2012.
- [30] Z. Zhao, "Positive solutions of nonlinear second order ordinary differential equations," *Proceedings of the American Mathematical Society*, vol. 121, no. 2, pp. 465–469, 1994.
- [31] J. Karamata, "Sur un mode de croissance régulière. Théorèmes fondamentaux," *Bulletin de la Société Mathématique de France*, vol. 61, pp. 55–62, 1933.
- [32] F.-C. Cirstea and V. Radulescu, "Uniqueness of the blow-up boundary solution of logistic equations with absorption," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 335, no. 5, pp. 447–452, 2002.
- [33] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation. Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1989.
- [34] V. Marić, *Regular Variation and Differential Equations*, vol. 1726 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.
- [35] E. Seneta, "Regularly varying functions," in *Lecture Notes in Math*, vol. 508, Springer-Verlag, Berlin, Germany, 1976.
- [36] R. Chemmam, H. Mâagli, S. Masmoudi, and M. Zribi, "Combined effects in nonlinear singular elliptic problems in a bounded domain," *Advances in Nonlinear Analysis*, vol. 1, no. 4, pp. 301–318, 2012.

