KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

MATH-244 (Linear Algebra); Final Exam; Semester 432

Max. Marks: 40 **Max. Time: 3 hours**

Note: ***Attempt all the five questions!***

Question 1 [3+2+2 marks]:

a) If $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then find $adj(adj(A))$.

b) Find the values of k that makes the matrix $\begin{bmatrix} 2 & 3k-2 \\ k^2 & -1 \end{bmatrix}$ symmetric.

c) Let $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 4 \\ 0 & 0 & 6 \end{bmatrix}$. Explain! Why the matrix B can be expressed as a product of elementary matrices?

Question 2 [4+3 marks]:

a) Solve of the linear system of equations with augmented matrix:

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ -1 & 2 & 0 & -2 & 2 \\ 3 & 1 & 0 & 3 & -1 \end{array}\right].$$

b) Solve the following linear system of equations by Cramer's Rule:

$$\begin{aligned} x - y \quad\quad &= 1 \\ -2x + 3y - 4z &= 0 \\ -2x + 3y - 3z &= 1 \end{aligned}$$

Question 3 [2+2+4 marks]:

a) Show that $E = \{ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 : a, b \in \mathbb{R}\}$ is a real vector space under usual addition and scalar multiplication of polynomials.

b) Find a basis and dimension of the vector space E.

c) Show that $<(x_1, x_2, x_3), (y_1, y_2, y_3)> = x_1y_1 + 2x_2y_2 + 3x_3y_3$ defines an inner product on the vector space $\mathbb{R}^3$. Then find an orthogonal basis of $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on $\{u_1 = (1,1,1),\ u_2 = (1,1,0),\ u_3 = (0,1,0)\}$.

Question 4 [3+3+3 marks]**:**

Let $\boldsymbol{B} = \{u_1 = (1,-1), u_2 = (1,1)\}$ and $\boldsymbol{C} = \{v_1 = (1,1,0), v_2 = (1,0,1),\ v_3 = (1,1,1)\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Let $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $\mathbf{T}(1,-1) = (3,5,2)$, $\mathbf{T}(1,1) = (2,-1,-3)$. Then find:

a) $\mathbf{T}(\mathbf{1},\mathbf{0})$ and $\mathbf{T}(\mathbf{0},\mathbf{1})$.

b) The matrix $[\mathbf{T}]_{\boldsymbol{B}}^{\boldsymbol{C}}$ of the linear transformation $\mathbf{T}$ with respect to the bases $\boldsymbol{B}$ and $\boldsymbol{C}$.

c) Coordinate vector $[\mathbf{T}(0,1)]_{\boldsymbol{C}}$ by using $[\mathbf{T}]_{\boldsymbol{B}}^{\boldsymbol{C}}$.

Question 5 [2+2+2+3 marks]**:**

Let $\boldsymbol{\lambda_1} = \mathbf{0}$, $\boldsymbol{\lambda_2} = \mathbf{1}$ and $\boldsymbol{\lambda_3} = -\mathbf{1}$ be the eigenvalues of 8×8 matrix $\boldsymbol{A}$ with algebraic multiplicities **3**, **2** and **3**, respectively. Let $\mathbf{dim}(\boldsymbol{E}_{\lambda_1}) = \mathbf{3}$, $\mathbf{dim}(\boldsymbol{E}_{\lambda_2}) = \mathbf{2}$ and $\mathbf{dim}(\boldsymbol{E}_{\lambda_3}) = \mathbf{3}$, where $\boldsymbol{E}_{\lambda_j}$ denotes the eigenspace with respect to the eigenvalue $\boldsymbol{\lambda_j}$.

a) Find the characteristic polynomial $\boldsymbol{q_A}(\boldsymbol{\lambda})$ of the matrix $\boldsymbol{A}$.

b) Explain, why the matrix $\boldsymbol{A}$ is diagonalizable?

c) Find the diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$, where $\boldsymbol{P}$ is an invertible matrix.

d) Find $\boldsymbol{A}^{\mathbf{11}}$.

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