King Saud University
Faculty of Sciences
Department of Mathematics

Final Examination	Math 481	Semester I - 1443	Time: 3H

Question 1 :

1. Prove that if f is Riemann integrable on an interval $[a,b]$, then $|f|$ is also Riemann integrable on $[a,b]$ and
$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

2. Consider the function $g\colon [0,1] \longrightarrow \mathbb{R}$ defined by:
$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ -1 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$
Prove that $|g|$ is Riemann integrable on $[0,1]$ but g is not Riemann integrable on $[0,1]$.

3. Determine if the following improper integrals are convergent or divergent:

   (a) $\displaystyle\int_1^{+\infty} \frac{1}{x+e^x}dx$ (b) $\displaystyle\int_1^{2} \frac{1}{x\ln^2 x}dx$

Question 2 :

Let $f_n(x) = \dfrac{1-x^n}{1+x^n}$, for $x \in [0,+\infty)$.

1. Study the convergence of the sequence $(f_n)_n$.

2. Study the uniform convergence of the sequence $(f_n)_n$ on the interval $[0,1]$.

Question 3 :

1. Prove that the series $\sum_{n=1}^{+\infty} \frac{x^n}{n\sqrt{n+1}}$ is uniformly convergent on $[-1,1]$.

2. Prove that the series $\sum_{n=1}^{+\infty} \frac{\sin(nx)}{\sqrt{n^5+n}}$ is uniformly convergent on $\mathbb{R}$.

Question 4 :

1. Prove that the Borel sigma-algebra is generated by the family $\mathscr{A} = \{[a,b],\ a \leq b \in \mathbb{R}\}$.

2. Prove that if f is measurable then $|f|$ is measurable.

Question 5 :

1. State the definition of a measurable set with respect to the Lebesgue outer measure m^* on $\mathscr{P}(\mathbb{R})$.

2. Prove that if $m^*(A) = 0$, then A is measurable.

3. State the definition of a measure on a $\sigma-$algebra $\mathscr{A}$.

4. Define $\mu\colon \mathscr{P}(\mathbb{R}) \longrightarrow \bar{\mathbb{R}}$ by:

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ countable} \\ \infty & \text{otherwise} \end{cases}$$

   Prove if μ is a measure on $\mathscr{P}(\mathbb{R})$.

5. Prove that $\mathbb{Q}$ is a Borel set in $\mathbb{R}$ and $m(\mathbb{Q}) = 0$, where m is the Lebesgue measure on $\mathbb{R}$.

Question 6 :

1. State the Boundedness Convergence Theorem.

2. Find with justification $\lim_{n\to+\infty} \int_0^1 \frac{nx}{1+n^2x^2}dx$.