Solution of the Final Examination   Math 481   Trimester III - 1444

Question 1 :

1. If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a differentiable function, then f is continuous and f is Riemann integrable on $[a,b]$.

2. The function $f = 1$ on $\mathbb{Q} \cap [a,b]$ and $f = -1$ on $\mathbb{Q}^c \cap [a,b]$ is not Riemann integrable on $[a,b]$, but $|f| = 1$ is Riemann integrable on $[a,b]$.

3. Let f a non-Riemann integrable function on $[a,b]$ and $g = -f$. Then $f + g = 0$ is Riemann integrable on $[a,b]$.

4. Any increasing function $f\colon [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable. If $\sigma = \{x_1, \ldots, x_n\}$ is a partition of $[a,b]$, then $0 \le U(f,\sigma) - L(f,\sigma) = \sum_{k=0}^{n-1}(x_{k+1} - x_k)(f(x_{k+1}) - f(x_k)) \le \|\sigma\|(f(b) - f(a)$. Then f is Riemann integrable on $[a,b]$.

Question 2 :

1. (a) Prove that $\int_a^b f(x)dx = 0 = \int_a^x f(t)dt + \int_x^b f(t)dt = 2\int_a^x f(t)dt$.

   Then $\int_a^x f(t)dt = 0$ for all $x \in [a,b]$.

   (b) If f is continuous, $\dfrac{d}{dx}\int_a^x f(t)dt = f(x) = 0$ then $f = 0$ on $[a,b]$.

Question 3 :

1. $\dfrac{1}{x^{\frac{1}{3}}(1+x)} \le \dfrac{1}{x^{\frac{1}{3}}}$ and $\dfrac{1}{x^{\frac{1}{3}}(1+x)} \le \dfrac{1}{x^{\frac{4}{3}}}$. Then the integral $\int_0^{+\infty} \dfrac{1}{x^{\frac{1}{3}}(1+x)}dx$ is convergent.

2. $\lim_{x\to-\infty} x^2 e^{x^3} = 0$, then the integral $\int_{-\infty}^{-1} e^{x^3}dx$ is convergent.

Question 4 :

1. $|\sin(\frac{x}{n^2})| \leq \frac{|x|}{n^2}$, then the series $\sum_{n\geq 1} \sin(\frac{x}{n^2})$ on $\mathbb{R}$.
   $\sup_{x\in\mathbb{R}} || \sin(\frac{x}{n^2})| = 1$, then the series $\sum_{n\geq 1} \sin(\frac{x}{n^2})$ is not uniformly convergent on $\mathbb{R}$.
2. (a) $\lim_{n\to+\infty} f_n(x) = 0$ for $x \neq 0$ and $f_n(0) = 0$.
   (b) $\sup_{x\in[0,+\infty)} |f_n(x)| = f_n(\frac{1}{\sqrt{n}})$. Then the sequence converges uniformly on $[0, +\infty)$.
   (c) As $|f_n(x)| \leq \frac{1}{2n^{\frac{3}{2}}}$, then the series $\sum_{n\geq 1} f_n$ on $[0, +\infty)$ is uniformly convergent ob $\mathbb{R}$.

Question 5 :

1. $\mu^*(A) = \inf\{\sum_{n=1}^{+\infty} \mathscr{L}(I_n) : \ I_n \text{ open}, \ A \subset \cup_{n=1}^{+\infty} I_n\}$.
2. We say that a function $f : \Omega \longrightarrow \mathbb{R}$ is measurable if $f^{-1}(A) \in \mathscr{B}$ for any Borel set A, $(A \in \mathscr{B}_{\mathbb{R}})$. This is equivalent that $f^{-1}[a,+\infty[\in \mathscr{B}$ for every $a \in \mathbb{R}$.
3. $\chi_E^{-1}[a,+\infty[= \emptyset$ if $a > 1$. $\chi_E^{-1}[a,+\infty[= E$ if $a \leq 1$. Then E of $\mathbb{R}$ is measurable if and only if the function χ_E is measurable.
4. If E is a null set and X a subset of $\mathbb{R}$. $m^*(X \cap E) \leq m^*(E) = 0$ and $m^*(X \cap E^c) \leq m^*(X)$. Then $m^*(X) \geq m^*(X \cap E^c) + m^*(X \cap E)$. Then E is measurable. $\left|\int_E f(x)dm(x)\right| \leq \infty m(E) = 0$ for all measurable function f on $\mathbb{R}$.

Question 6 :

1. The monotone convergence theorem:
   Let $(f_n)_n$ be an increasing sequence of non-negative measurable functions on Ω, then

$$\int_\Omega \lim_{n\to+\infty} f_n(x)d\lambda(x) = \lim_{n\to+\infty} \int_\Omega f_n(x)d\lambda(x).$$

The Dominate Convergence Theorem

Let $(f_n)_n \in \mathscr{M}(\Omega)$ such that

(a) $(f_n)_n$ converges a.e. to a function f defined a.e.

(b) There exists a non negative integrable function g so that: $|f_n| \leq g$ a.e. for every n.
Then the sequence $(f_n)_n$ and the function f is integrable and

$$\int_\Omega f(x)\, d\,\lambda(x) \;=\; \lim_{n\to+\infty} \int_\Omega f_n(x) d\,\lambda(x).$$

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2. $If f_n(x) = \dfrac{nx}{1+n^3x^3}$, then $f_n(x) \leq f_n(\dfrac{1}{2^{\frac{1}{3}}n} = \leq 1$. Then $\displaystyle\lim_{n\longrightarrow+\infty} \int_0^1 \frac{nx}{1+n^3x^3} dx =$ 0.