King Saud University
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Department of Mathematics

Solution of Final Examination Math 481 Semester I - 1445

Question 1 :

1. (a) Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then for all $[a,b] \subset [0,1]$, $\sup\limits_{x\in[a,b]} f(x) = b$ and $\inf\limits_{x\in[a,b]} f(x) = -b$. We deduce that $U(f) = \int_0^1 x dx = \frac{1}{2}$ and $L(f) = \int_0^1 -x dx = -\frac{1}{2}$.

   (b) As $U(f) \neq L(f)$, then f is not Riemann integrable.

   (c) $f(x) = x$ a.e, then f Lebesgue integrable and $\int_{[0,1]} f(x)dm(x) = \frac{1}{2}$.

2.

$$\begin{aligned} S(f, P_n, \alpha_n) &= \frac{1}{n}\sum_{k=1}^{n}(\frac{k^2}{n^2} - \pi\frac{k}{n}) \\ &= \frac{1}{n^3}\sum_{k=1}^{n} k^2 - \frac{\pi}{n^2}\sum_{k=1}^{n} k \\ &= \frac{n(n+1)(2n+1)}{6n^3} - \frac{\pi n(n+1)}{2n^2}. \end{aligned}$$

Then $\int_0^1 (x^2-\pi x)dx = \lim\limits_{n\to+\infty}\left(\frac{n(n+1)(2n+1)}{6n^3} - \frac{\pi n(n+1)}{2n^2}\right) = \frac{1}{3} - \frac{\pi}{2}$.

Question 2 :

1. $\lim\limits_{n\to+\infty} f_n(x) = 0$, for all $x \in \mathbb{R}$.

2. $f_n(\frac{1}{n}) = \frac{1}{2}$, then the sequence $(f_n)_n$ is not uniformly convergent on $[0,1]$. For all $n \in \mathbb{N}$, the function f_n is decreasing on the interval $[1,2]$, then $0 \leq f_n(x) \leq f_n(1) = \frac{n}{1+n^2}$ and the sequence $(f_n)_n$ is uniformly convergent on the interval $[1,2]$.

3. Course

4. On the interval $[0,1]$, $0 \leq f_n(x) \leq \frac{1}{2}$, which is integrable on $[0,1]$, then by dominate convergence theorem $\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}dx = 0$.

Question 3 :

1. Since the sequence $\left(\frac{1}{k+x^2}\right)_n$ is decreasing then $\left|\sum_{k=n}^{m} \frac{(-1)^k}{k+x^2}\right| \leq \frac{1}{n+x^2}$, for all $n \leq m \in \mathbb{N}$ and $x \in \mathbb{R}$.

2. (a) $\sup_{x\in\mathbb{R}} \left|\sum_{k=n}^{m} \frac{(-1)^k}{k+x^2}\right| \leq \frac{1}{n}$, then the series $\sum_{n\geq 1} f_n(x)$ is uniformly convergent on $\mathbb{R}$.

   (b) Since the functions $f_n(x) = \frac{(-1)^n}{n+x^2}$ are continuous and the convergence of the series $\sum_{n\geq 1} f_n(x)$ is uniform, then the function f is continuous on $\mathbb{R}$.

   (c) Since the convergence of the series $\sum_{n\geq 1} f_n(x)$ is uniform on $\mathbb{R}$ and
   $$\lim_{x\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k+x^2} = 0, \text{ then } \lim_{x\to+\infty} f(x) = 0.$$

Question 4 :

1. A subset E of $\mathbb{R}$ is said to be measurable with respect to the Lebesgue outer measure m^* if
$$m^*(X) = m^*(X \cap E) + m^*(X \cap E^c), \quad \forall X \subset \mathbb{R}.$$

2. If $m^*(E) = 0$, then for $X \subset \mathbb{R}$, $m^*(X\cap E) = 0$ and $m^*(X\cap E^c) \leq m^*(X)$. Then $m^*(X) \geq m^*(X \cap E) + m^*(X \cap E^c)$. Moreover as m^* is an outer measure, $m^*(X) \leq m^*(X \cap E) + m^*(X \cap E^c)$. Then $m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$ and E is measurable.

3. As $m^*(\{a\}) = 0$ for all $a \in \mathbb{R}$, then for any countable set E in $\mathbb{R}$, $m^*(E) = 0$.

4. As $m^*([0,1]) = 1$, then $[0,1]$ is not countable.

Question 5 :

1. For $a > 0$, $(\frac{1}{f})^{-1}([a,+\infty]) = f^{-1}([0,\frac{1}{a}])$, which is measurable. If $a = 0$, $(\frac{1}{f})^{-1}([0,+\infty]) = f^{-1}([0,+\infty])$, which is measurable. For $a < 0$, $(\frac{1}{f})^{-1}([a,+\infty]) = f^{-1}((0,+\infty])$, which is measurable.
2. Course.
3. $\lim\limits_{n\to+\infty} \chi_{[0,n]}(1+\frac{x}{n})^n e^{-2x} = e^{-x}$. $\chi_{[0,n]}(1+\frac{x}{n})^n e^{-2x} \leq e^{-x}$. Then by dominate convergence theorem,
$$\lim_{n\to+\infty} \int_0^n (1+\frac{x}{n})^n e^{-2x} dx = \int_0^{+\infty} e^{-x} dx = 1.$$
4. $\dfrac{(x\ln x)^2}{1+x^2} = \sum\limits_{n=0}^{+\infty} (-1)^n x^{2n+2} \ln^2 x$, $\left|\sum\limits_{k=0}^{n} (-1)^k x^{2k+2} \ln^2 x\right| \leq x^2 \ln^2 x$ which is integrable on $[0,1]$. Then
$$\begin{aligned}\int_0^1 \frac{(x\ln x)^2}{1+x^2} &= \sum_{n=0}^{+\infty} (-1)^n \int_0^1 x^{2n+2} \ln^2 x dx \\ &= 2\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n+1)^3}.\end{aligned}$$