King Saud University
Faculty of Sciences
Department of Mathematics

Final Examination	Math 481	Semester I - 1445	Time: 3H

Question 1 :

1. Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0,1] \\ -x & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

   (a) Find $U(f)$ and $L(f)$.

   (b) Is f Riemann integrable?

   (c) Is f Lebesgue integrable?

2. Use Darboux sums $S(f, P_n, \alpha_n)$ to compute the integral $\int_0^1 (x^2 - \pi x)dx$. ($P_n = \{x_0, \ldots, x_n\}$ the uniform partition of $[0,1]$ and α_n the mark on P_n defined by: $\alpha_n = (x_1, \ldots, x_n)$).

Question 2 :

Define the sequence of functions $(f_n)_n$ on $\mathbb{R}$ by: $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for all $n \in \mathbb{N}$.

1. Prove that the sequence $(f_n)_n$ is convergent and find its limit.

2. Show that the sequence $(f_n)_n$ is uniformly convergent on the interval $[1,2]$ but is not uniformly convergent on $[0,1]$.

3. State the Bounded Convergence Theorem.

4. Evaluate $\lim\limits_{n\to\infty} \int_0^1 \dfrac{nx}{1+n^2x^2}dx$.

Question 3 :

Define the sequence of functions $(f_n)_n$ on $\mathbb{R}$ by: $f_n(x) = \dfrac{(-1)^n}{n+x^2}$, for all $n \in \mathbb{N}$.

1. Prove the inequality $\left|\sum_{k=n}^{m} \frac{(-1)^k}{k+x^2}\right| \leq \frac{1}{n+x^2}$, for all $n \leq m \in \mathbb{N}$ and $x \in \mathbb{R}$.

2. Deduce

   (a) the series $\sum_{n\geq 1} f_n(x)$ is uniformly convergent on $\mathbb{R}$,

   (b) the function f is continuous on $\mathbb{R}$, where $f(x) = \sum_{n=1}^{+\infty} f_n(x)$,

   (c) $\lim_{x\to+\infty} f(x) = 0$.

Question 4 :

1. State the definition of a measurable set with respect to the Lebesgue outer measure m^*.

2. Prove that if $m^*(E) = 0$, then E is measurable.

3. Prove that $m^*(E) = 0$ for any countable set E in $\mathbb{R}$.

4. Deduce that $[0, 1]$ is not countable.

Question 5 :

1. Prove that if $f : \mathbb{R} \longrightarrow (0, +\infty]$ is Lebesgue measurable, then $\frac{1}{f}$ is also Lebesgue measurable.

2. State Fatou lemma.

3. Use Lebesgue integral theorems to compute the following limit

$$\lim_{n\to+\infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx.$$

4. Prove that $\int_0^1 \frac{(x \ln x)^2}{1+x^2} = 2\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n+1)^3}$.

   (Hint: use the power series representation $\frac{1}{1+x^2} = \sum_{n=0}^{+\infty}(-1)^n x^{2n}$.)