Q.1 (4) $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{n^2}$, $a_n=\left(1+\frac{2}{n}\right)^{n^2}$, $n\geq 1$, A positive term series

Root: $\sqrt[n]{a_n}=\left[\left(1+\frac{2}{n}\right)^{n^2}\right]^{1/n}=\left(1+\frac{2}{n}\right)^{n}=\left[\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^2$

(2)

$\lim_{n\to\infty}\sqrt[n]{a_n}=\left[\lim_{\frac{n}{2}\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{n/2}\right]^2=e^2>1$

$\therefore$ By root test, the given series diverges. (2)

Q.2 we know that

(5) $\frac{1}{1+x}=1-x+x^2-x^3+x^4-\cdots+(-1)^n.x^n+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}.x^{n-1}$, $|x|<1$ (1)

Replacing x with x^2, we get

$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n.x^{2n}$, $x^2<1 \Rightarrow |x|<1$ (1/2)

Integrating both sides w.r.t. x, we get

$\int\frac{dx}{1+x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n.x^{2n+1}}{2n+1}$ —(i) (1)

$\Rightarrow f(x)=\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n.x^{2n+1}}{2n+1}$, this is the required power series representation of $f(x)=\tan^{-1}x$.

Radius of Convergence $=1$ (1/2)

For Interval of Convergence, we need to investigate behavior of the Series in (i) at $x=\pm 1$.

At $x=1$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ which is a convergent A.S.

At $x=-1$, $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ which is again a convergent A.S. (1)

$\therefore$ IoC $=[-1,1]$.

Let $x = \frac{1}{\sqrt{3}} \in [-1,1]$ in (i), then

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{-n-1/2}}{(2n+1)} \Rightarrow \frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cdot \frac{1}{3^{n+1/2}}$$

(1)

Q.3

(4)

$P_1: \quad 3x + 12y - 6z = -2 \quad$ — (i)

$P_2: \quad 5x + 20y - 10z = 7 \quad$ — (ii)

$\because \quad \frac{3}{5} = \frac{12}{20} = \frac{-6}{-10} \Rightarrow P_1 \parallel P_2$ (1)

Let $x=0, y=0$, then $P(0,0,1/3)$ is any point on (i).

Distance of P from P_2 is given by (1)

$$d = \frac{|5(0) + 20(0) - 10\cdot\left(\frac{1}{3}\right) - 7|}{\sqrt{25+400+100}} = \frac{31}{3\sqrt{525}} \text{ units}$$

$\simeq 0.45$ units. (2)

Q.4 We know that

(3)

$$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \qquad \because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + 2\,\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$ (1)

Now $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 \iff \vec{a}\cdot\vec{b} = 0$

$\iff \vec{a} \perp \vec{b}$. (2)

i.e, given equality will be true iff $\vec{a}$ & $\vec{b}$ are orthogonal

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Q.5 (4) $S: y = -\frac{x^2}{16} - \frac{z^2}{4}$, it is a paraboloid having axis along y-axis, vertex at (0,0,0) and opening along -ve y-axis. (1)

xy-Trace: Put $z=0$, then $y = -\frac{x^2}{16}$, parabolas facing along -ve y-axis. (1)

xz-Trace: Put $y=0$, then $\frac{x^2}{16} + \frac{z^2}{4} = 0 \Rightarrow x=0, z=0$

$\therefore$ xz-Trace is the origin (0,0,0). (1)

yz-Trace: Put $x=0$, then $y = -\frac{z^2}{4}$, parabolas facing along -ve y-axis. (1)

Q.6 (5) Let $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 2 = 0$ — (i)

Direction of max ↑ in f is $\nabla f = f_x \hat{i} + f_y \hat{j}$

At (0,0), (i) $\Rightarrow z = \pm\sqrt{\frac{2}{3}}$

$f_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(2x)}{6z} = -\frac{x}{3z}$, $f_x(0,0) = 0$ (1)

$f_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2}{6z} = -\frac{1}{3z}$, $f_y(0,0) = \pm\sqrt{\frac{3}{2}}$ (1)

$\therefore \nabla f(0,0) = \pm\sqrt{\frac{3}{2}}\,\hat{j}$

A unit vector in the direction of $\nabla f(0,0)$ is $\hat{u} = \pm\hat{j}$ (1)

$\therefore D_u f(0,0) = \nabla f(0,0) \cdot \hat{u} = \sqrt{\frac{3}{2}}$ (2)

Q.7 (5)

$f(x,y) = \frac{x^2}{2} + 2xy - \frac{y^2}{2} - 8y + x$ —(i)

$f_x = x + 2y + 1, \quad f_y = 2x - y - 8$

For critical pts. of (i), $f_x = 0$ & $f_y = 0$

$\Rightarrow \quad x + 2y = -1$ —(ii)

$2x - y = 8$ —(iii)

(ii) + 2(iii) $\Rightarrow$

$x + 2y = -1$

$4x - 2y = 16$

$5x = 15$

$\Rightarrow x = 3$

$\therefore$ (iii) $\Rightarrow y = 2x - 8 = -2$

$\therefore (3,-2)$ is a critical pt. (1)

For Disc: $f_{xx} = 1, \; f_{yy} = 2, \; f_{xy} = 2 = f_{yx}$

$\therefore D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2-4 = -2 < 0$

$\because D < 0, \forall (x,y)$, $\therefore (3,-2)$ is a Saddle pt. (1)

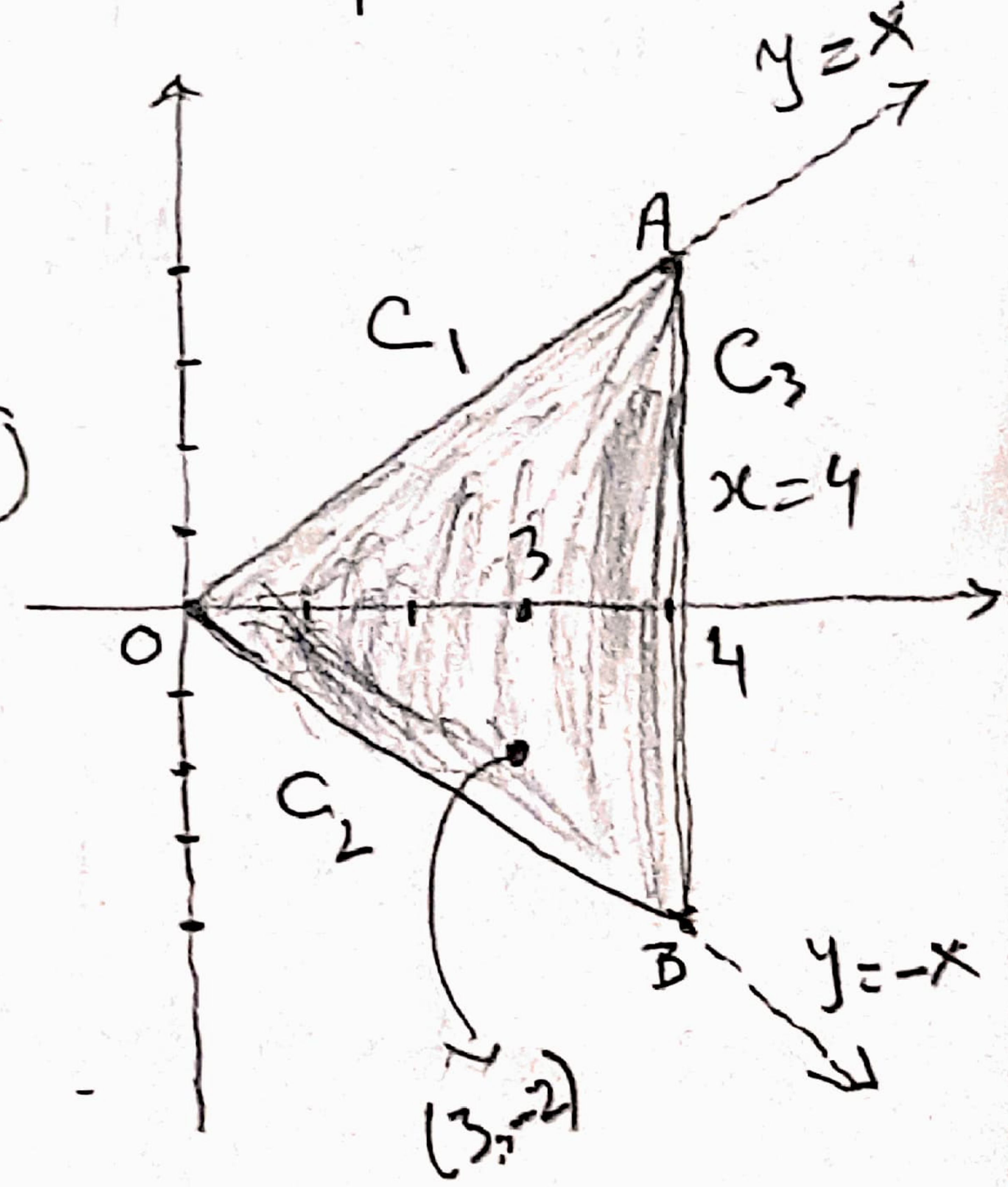
B'dary Extrema:

R consists of C_1, C_2 & C_3.

$C_1: \{(x,y): y = x, 0 \le x \le 4\}$

$C_2: \{(x,y): y = -x, 0 \le x \le 4\}$

$C_3: \{(x,y): x = 4, -4 \le y \le 4\}$ (1)



Along C_1: $y = x$

$h(x) = \frac{x^2}{2} + 2x^2 - \frac{x^2}{2} - 8x + x = 2x^2 - 7x$

$\Rightarrow h(x) = x(2x-7), \; x \in [0,4]$

$h'(x) = 4x - 7 = 0 \Rightarrow x = \frac{7}{4} \in [0,4]$ is a critical pt.

$h''(x) = 4 > 0 \Rightarrow h(x)$ has a min. at $x = 7/4$

min. value $= h(7/4) = \frac{7}{4} \times \left(\frac{7}{2} - 7\right) = \frac{7}{4} \times \frac{-7}{2} = -\frac{49}{8}$ ✓ (1)

End Point Extrema:

At $(0,0)$ & $(4,4)$, $h(x)$ has maxima, given by

$h(0) = 0$ & $h(4) = 4\times(8-7) = 4$

Along C_2: $y = -x$

(i) $\Rightarrow$ $g(x) = \frac{x^2}{2} - 2x^2 - \frac{x^2}{2} + 8x + x = -2x^2 + 9x$, $x \in [0,4]$

$g'(x) = -4x + 9 = 0 \Rightarrow x = \frac{9}{4} \in [0,4]$

$g''(x) = -4 < 0$, $\Rightarrow$ 'g' has a max. at $x = 9/4$.

Max. value $= g(9/4) = -2\times\frac{81}{16} + \frac{81}{4} = -\frac{81}{8} + \frac{81}{4} = \frac{81}{8}$ ✓ (1)

End point Extrema:

At $x=4$, g has a min. & min. value $= g(4) = -2\times16 + 36 = 4$

Along C_3: $(x=4)$

$k(y) = 8 + 8y - \frac{y^2}{2} - 8y + 4 = -\frac{y^2}{2} + 12$, $y \in [-4,4]$

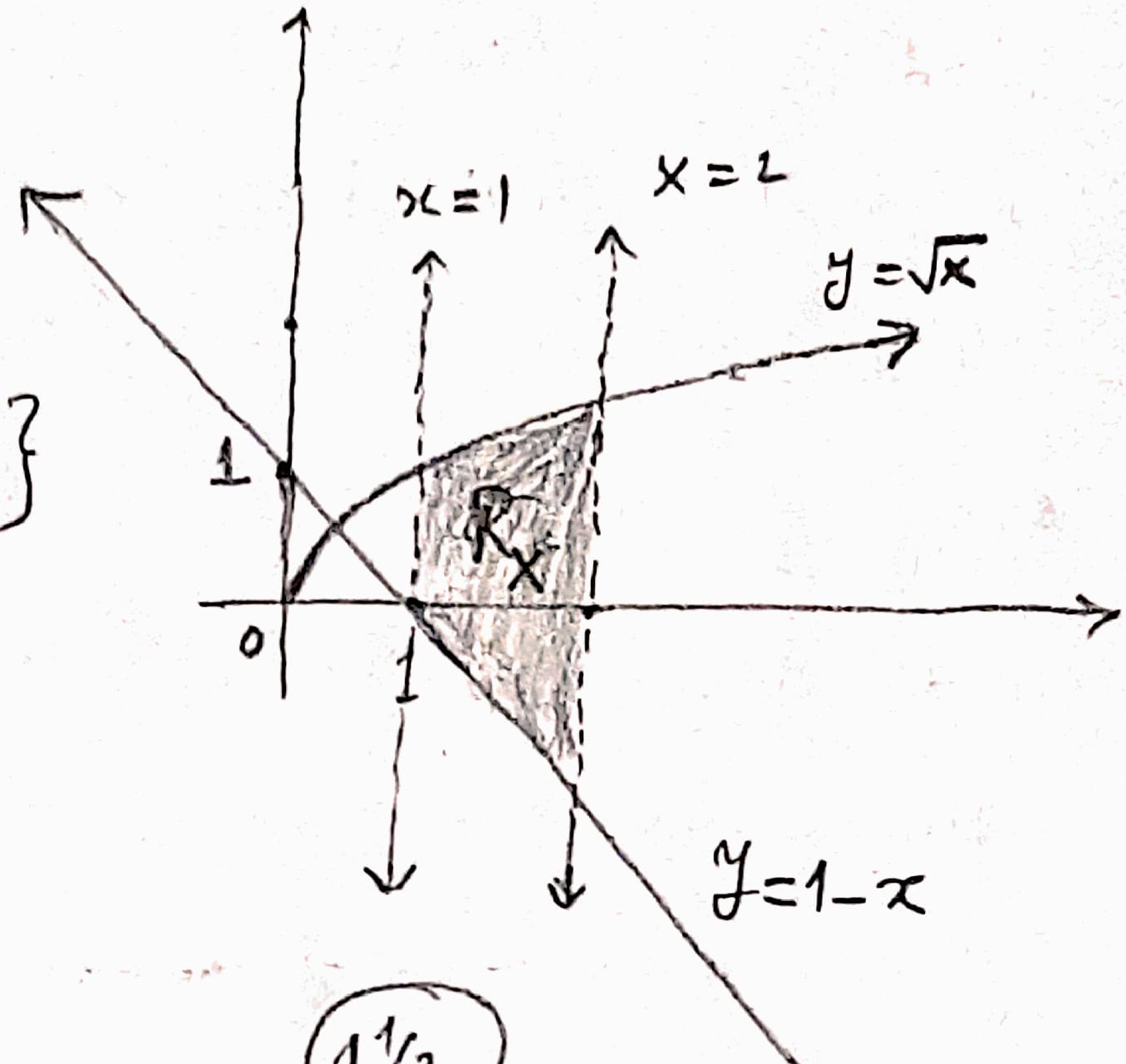
$k'(y) = -y = 0 \Rightarrow y = 0$

$k''(y) = -1 < 0 \Rightarrow k(y)$ has a max. at $y = 0$.

Max. value $= k(0) = 12$. ✓ (1)

Q.8 (4) The region R is shown in the figure. It is an R_x region,

$R_x = \{(x,y) : 1-x \le y \le \sqrt{x},\ 1 \le x \le 2\}$



$\therefore \iint_R x^2 y \, dA = \int_1^2 \int_{1-x}^{\sqrt{x}} x^2 y \, dy \, dx$ (1½)

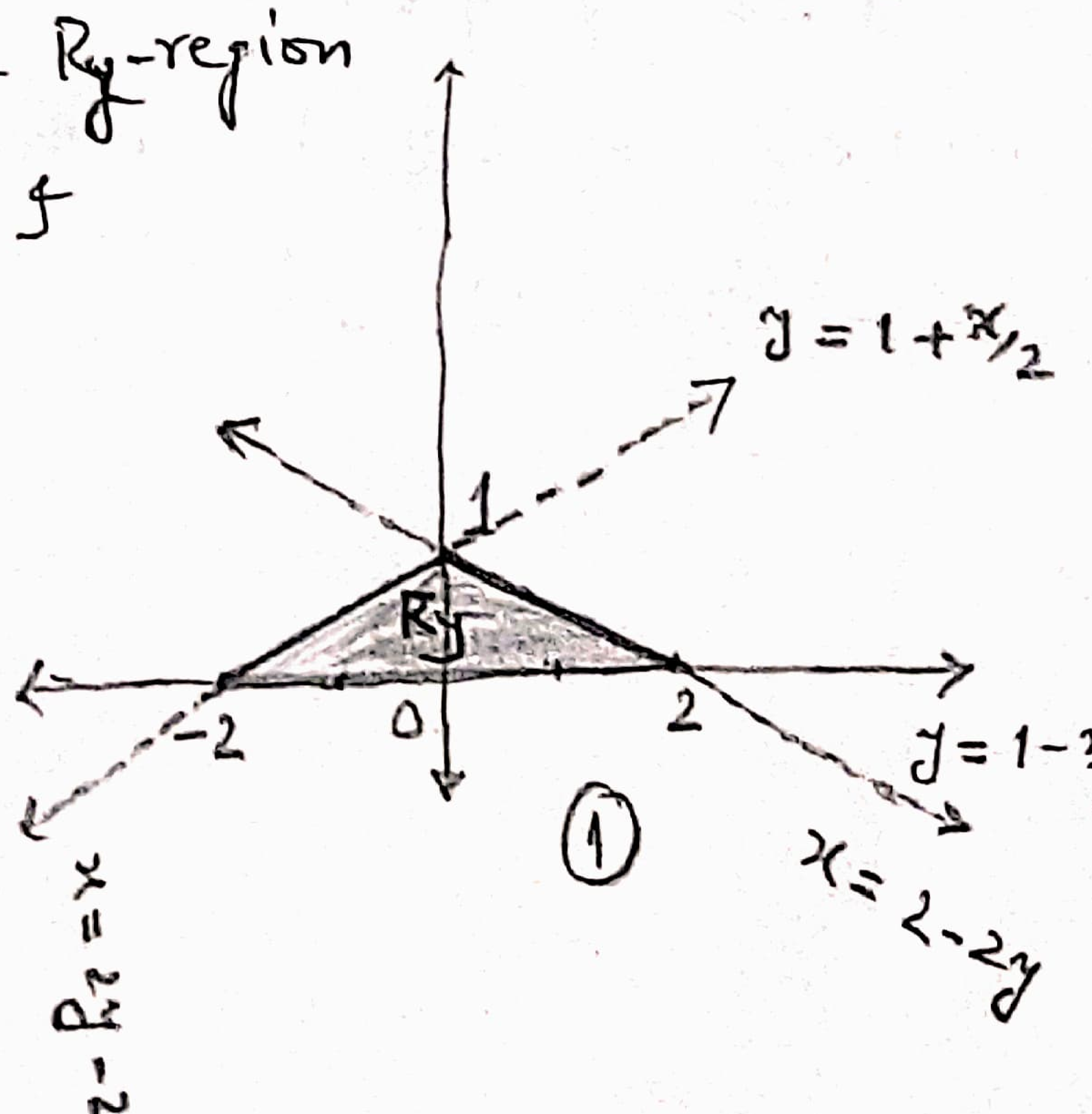
$= \int_1^2 x^2 \left| \frac{y^2}{2} \right|_{1-x}^{\sqrt{x}} dx$

$= \int_1^2 x^2 \left[\frac{x}{2} - \frac{(1-x)^2}{2} \right] dx = \frac{1}{2} \int_1^2 (x^3 - x^2 - x^4 + 2x^3) \, dx$

$= \frac{1}{2} \left| \frac{3x^4}{4} - \frac{x^3}{3} - \frac{x^5}{5} \right|_1^2 = \frac{1}{2} \left[12 - \frac{8}{3} - \frac{32}{5} - \frac{3}{4} + \frac{1}{3} + \frac{1}{5} \right]$

$= \frac{1}{2} \left[12 - \frac{7}{3} - \frac{31}{5} - \frac{3}{4} \right] = \frac{1}{2} \cdot \frac{720 - 140 - 372 - 45}{60} = \frac{163}{120} \approx 1.4$ (1)

Q.9 Base of the solid is an R_y-region with left b'dary $x = 2y-2$ & right b'dary $x = 2-2y$.



$\therefore V = \iint_{R_y} (x^2 + 4y^2) \, dA$ (2)

$= \int_0^1 \int_{2y-2}^{2-2y} (x^2 + 4y^2) \, dx \, dy$

$= \int_0^1 \left| \frac{x^3}{3} + 4xy^2 \right|_{2y-2}^{2-2y} dy$

$= \int_0^1 \left[\frac{(2-2y)^3}{3} + 4y^2(2-2y) - \frac{(2y-2)^3}{3} - 4y^2(2y-2) \right] dy$ (2)

$= \left[-\frac{1}{6} \frac{(2-2y)^4}{4} + 8 \cdot \frac{y^3}{3} - 8\frac{y^4}{4} + \frac{1}{6} \frac{(2y-2)^4}{4} - 8 \cdot \frac{y^4}{4} + 8\frac{y^3}{3} \right]_0^1 = 0 + \frac{8}{3} - 2 - 0 - 2 + \frac{8}{3} + \frac{2}{3} + \frac{2}{3}$

$= \frac{20}{3} - 4 = \frac{8}{3}$ cubic units ✓