Math 316

Final Exam 1434/1435, 1st semester
Exam Time: 3 hours

Q1 (a) Consider the Laguerre equation

$$xu'' + (1-x)\,u' + nu = 0, \quad x \in (0,\infty)\,, \quad n \in \mathbb{N}_0 \tag{1}$$

i. Put equation (1) in the standard form: $Lu + \lambda u = 0$. [1 Mark]

ii. Show that the differential operator L is not formally self-adjoint, then transform it to a formally self-adjoint operator. [3 Marks]

iii. Laguerre polynomials $\{\mathbf{L}_n, n \in \mathbb{N}_0\}$ are eigenfunctions of equation (1), write the orthogonality relation. [1 Mark]

(b) Solve the heat equation

$$u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0,$$

with the following boundary and initial conditions

$$\begin{aligned} u(0,t) &= u(\pi,t) = 0, \qquad t > 0, \\ u(x,0) &= f(x)\,, \qquad 0 < x < \pi. \end{aligned}$$

[5 Marks]

Q2 (a) Let f be piecewise smooth function on $[-\pi,\pi]$, periodic in 2π and $f(-\pi) = f(\pi)$. Show that the Fourier series of the function f can be integrated term-wise on any bounded interval I. [2 Marks]

(b) Consider the function

$$f(x) = |x|\,, \quad x \in [-\pi,\pi]$$

and

$$f(x+2\pi) = f(x)\,,\ x \in \mathbb{R}.$$

i. Find the Fourier series representation for f. [4 Marks]

ii. Show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$$

[1.5 Marks]

iii. Show that

$$x^2 = \pi x - \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\,x, \quad x \geq 0$$

(Hint: use part (a)). [2.5 Marks]

Q3 (a) Consider the identity

$$(n+1)\,P_{n+1}(x) + nP_{n-1}(x) = (2n+1)\,xP_n(x)\,, \quad x \in [-1,1]\,, \quad n \in \mathbb{N}.$$

where P_n is Legendre polynomial.

i. Show that

$$\begin{aligned} n\left\|P_{n-1}\right\|^2 &= (2n+1)\left\langle xP_n, P_{n-1}\right\rangle, \\ n\left\|P_n\right\|^2 &= (2n-1)\left\langle xP_{n-1}, P_n\right\rangle \end{aligned}$$

[2 Marks]

ii. Use part (i) to prove that

$$\left\|P_n\right\|^2 = \frac{2}{2n+1}.$$

[2 Marks]

(b) Consider the Fourier-Hermite representation of the function

$$f(x) = x, \quad x \in \mathbb{R}$$

Show that the Fourier-Hermite coefficient c_n is zero for all <u>even numbers</u> n in $\mathbb{N}_0$. [2 Marks]

(c) Find the first two Bessel functions of the first kind, J_0, J_1 and show that

$$J_0^{'}(x) = -J_1(x)\,.$$

[3 Marks]

Q4 Consider the function

$$f(x) = e^{-|x|}, \quad x \in \mathbb{R},$$

(a) Show that $f \in \mathcal{L}^1(\mathbb{R})$. [2 Marks]

(b) Find the Fourier transform of f . [4 Marks]

(c) Find the Fourier integral of f. [4 Marks]

Good Luck
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