King Saud University
Department Of Mathematics
M-203
(Differential and Integral Calculus)

First Mid-Term Examination
(II-Semester 1433/34)

Max. Marks: 25 Time: 90 Minutes

Q. No: 1 Determine whether or not the sequence $\left\{\left(1-\frac{2}{n}\right)^n\right\}_{n=1}^{\infty}$ converges, and if it converges find its limit..[4]

Q. No: 2 Find the sum of the series $\sum_{n=1}^{\infty}\left[\frac{5}{n(n+1)}+\frac{1}{2^n}\right]$..............................[5]

Q. No: 3 Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty}\frac{n}{e^{n^2+1}}$$ ....[5]

Q, No: 4 Find the power series representation for the function $f(x)=\frac{1}{3x-5}$ centered at $c=+2$. ....[5]

Q. No: 5 Find the interval of convergence and radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$ ....[6]

M-203

I Mid-term Exam. (II sem. 1433/1434)

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Q#1. Determine whether or not the sequence $\left\{\left(1-\frac{2}{n}\right)^n\right\}_{n=1}^{\infty}$ converges, and if it converges, find its limit.

[[illegible] Mark]

Soln We have $a_n = \left(1-\frac{2}{n}\right)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n \quad (1^\infty \text{ form})$$

Let $y = \left(1-\frac{2}{x}\right)^x$.

Then $\ln y = x\ln\left(1-\frac{2}{x}\right)$ $(0\cdot\infty \text{ form})$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(1-\frac{2}{x}\right) \quad 1$$

$$= \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0} \text{ form}\right)$$

Apply L'Hospital rule: $\lim_{x\to\infty}\dfrac{\frac{1}{1-\frac{2}{x}}\left(\frac{2}{\cancel{x^2}}\right)}{\left(-\frac{1}{\cancel{x^2}}\right)} = -2$ 1

$$\lim_{x\to\infty}\ln y = -2$$

$\Rightarrow \lim_{x\to\infty} y = e^{-2}$ 1 $\therefore \lim_{x\to\infty} y = \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x = e^{-2}$

Hence the given sequence is convergent and its limit is e^{-2} 1.

Q#2) Find the sum of the series: $\sum_{n=1}^{\infty}\left[\frac{5}{n(n+1)}+\frac{1}{2^n}\right]$

[Mark: 5]

Soln. $\sum_{n=1}^{\infty}\frac{5}{n(n+1)} = 5\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

$= 5\sum\left[\frac{1}{n}-\frac{1}{n+1}\right]$

$= 5\left[\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$

$= 5\left[1-\frac{1}{n+1}\right]$

$= 5\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right] = 5(1) = 5$ ✓ 2

Also, $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a conv. Geom. series with $a=\frac{1}{2}$, $r=\frac{1}{2}$

$\therefore$ Sum $= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ 2

Hence, $\sum_{n=1}^{\infty}\left[\frac{5}{n(n+1)}+\frac{1}{2^n}\right] = 5+1 = \underline{\underline{6}}$ 1 ✓

Q#3) Determine whether the following series Convg. or divgs.

$\sum_{n=1}^{\infty}\frac{n}{e^{n^2+1}}$

[Mark: 5]

Soln. $f(x) = \frac{x}{e^{x^2+1}} \Rightarrow f'(x) = \frac{e^{x^2+1}-xe^{x^2+1}(2x)}{\left(e^{x^2+1}\right)^2} = \frac{e^{x^2+1}(1-2x^2)}{\left(e^{x^2+1}\right)^2} < 0 \ \forall x \geq 1$

and f is clearly continuous on $[1,\infty)$ ✓ 2 $\Rightarrow f \downarrow$

Now, $\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{e^{x^2+1}}\,dx = \lim_{t\to\infty} -\frac{1}{2}\left[e^{-(x^2+1)}\right]_1^t$

$= -\frac{1}{2}\left[e^{-t^2+1} - e^{-2}\right] = \frac{1}{2e^2}$ 2 ✓

Hence, $\sum_{n=1}^{\infty}\frac{n}{e^{n^2+1}}$ is conv. 1

(3)

Q#4) Find the power series representation for the function $f(x) = \dfrac{1}{3x-5}$ centred at $c = 2$.

[Mark: 5]

Soln. We have $f(x) = \dfrac{1}{3x-5}$

$$= \frac{1}{3(x-2)+1}$$

$$= \frac{1}{1+3(x-2)}$$ (2) if $3|x-2| < 1$

$$= 1 - 3(x-2) + 3^2(x-2)^2 - 3^3(x-2)^3 + \cdots + (-1)^n 3^n (x-2)^n + \cdots$$

(3)

Q#5) Find the interval of convergence and radius of conv. of the power series $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$ [Mark: 6]

Soln. By absolute Ratio test, we have

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}} \times \frac{\sqrt{n+3}}{2^n (x-3)^n} \right|$$

$$= 2\,|x-3| \quad \text{as } \lim_{n\to\infty} \frac{\sqrt{n+3}}{\sqrt{n+4}} = 1 \quad ②$$

For abs. conv. we have $2\,|x-3| < 1$

$$\Leftrightarrow |x-3| < \tfrac{1}{2}$$

$$\Leftrightarrow -\tfrac{1}{2} < x-3 < \tfrac{1}{2}$$

$$\Leftrightarrow +\tfrac{5}{2} < x < \tfrac{7}{2} \quad ①$$

At $x = \frac{5}{2}$, we have $\sum_{n=0}^{\infty} \frac{2^n (\frac{5}{2}-3)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{\cancel{2^n}(-\frac{1}{\cancel{2}})^n}{\sqrt{n+3}}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+3}}$$

which is conv. by AST ①

At $x = \frac{7}{2}$, we have $\sum_{n=0}^{\infty} \frac{2^n (\frac{7}{2}-3)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{\cancel{2^n}(\frac{1}{\cancel{2}})^n}{\sqrt{n+3}}$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$$

Comparing with $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{\frac{1}{2}}}$ which is divg. by p-series

$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ is divg. ①

Hence interval of conv. $\left[\frac{5}{2}, \frac{7}{2}\right)$

Radius of conv. $r = \frac{1}{2}\left[\frac{7}{2} - \frac{5}{2}\right] = \frac{1}{2}$ ①