King Saud University
Department Of Mathematics.
M-203 (First Mid-Term)
(Differential and Integral Calculus)

(Summer Semester 1433/1434)

Max. Marks: 25 Time: 1.30 hrs

Marking Scheme:	**All questions carry equal marks.**

Q. No: 1 Discuss the convergence of the sequence $\left\{\frac{1}{n}\sin\left(\frac{1}{n^2}\right)\right\}$.

Q. No: 2 Find the sum of the series $\sum_{n=1}^{\infty}\left[\frac{1}{(n+1)(n+2)}+\frac{1}{e^n}\right]$.

Q. No: 3 Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=1}^{\infty}(-2)^n\frac{1}{\sqrt{n}}(x+3)^n$.

Q . No:4 Determine whether the series is **absolutely convergent, conditionally convergent**, or **divergent** $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)}$.

Q. No:5 Find the Taylor series for the function $f(x)=\cos x$ **at** $c=\pi$.

M-203

I Mid-term Exam. (Summer Semester 1433/1434)

Max. Marks: 25 Time: 90 Mts.

Q#1) Discuss the convergence of the sequence $\left\{\frac{1}{n}\sin\left(\frac{1}{n^2}\right)\right\}$. [Marks: 5]

Soln. we know $-1 \le \sin n \le 1$

Hence, we have $-1 \le \sin\left(\frac{1}{n^2}\right) \le 1$ ②

$$\Rightarrow \underbrace{\lim_{n\to\infty}(-1)\cdot\frac{1}{n}}_{=0} \le \lim_{n\to\infty}\frac{1}{n}\sin\left(\frac{1}{n^2}\right) \le \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Hence by Sandwich theorem $\lim_{n\to\infty}\frac{1}{n}\sin\frac{1}{n} = 0$, ② Conv. ①

Q#2) Find the sum of the series $\sum_{n=1}^{\infty}\left[\frac{1}{(n+1)(n+2)} + \frac{1}{e^n}\right]$ [Marks: 5]

Soln. we have $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ and $\sum_{n=1}^{\infty}\frac{1}{e^n}$

$$= \frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n} + \cdots$$

which is a conv. Geom Series with $r = \frac{1}{e}$.

$$\therefore \text{Sum } S_2 = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e}\left(\frac{e}{e-1}\right) = \frac{1}{e-1} = \frac{1}{2.71828-1} = \frac{1}{.71828}$$ ②

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

or $$\frac{1}{(n+1)(n+2)} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$$

Put $n=-1$, we have $1 = A$

Put $n=-2$, we have $1 = -B \therefore B = -1$

Hence, $$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{n}\frac{1}{(n+1)(n+2)} = \sum_{1}^{n}\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \left[\left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+1}\right)\right]$$

$$= \left[\frac{1}{2} - \frac{1}{n+2}\right]$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} = S_1 = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{n+2}\right] = \frac{1}{2}$$ ②

$\therefore$ Sum $S = S_1 + S_2$ ①

$$= \frac{1}{2} + \frac{1}{.71828}$$

Q#3) Find the Interval of Convergence and radius of conv. of the Power Series $\sum_{n=1}^{\infty} (-2)^n \frac{1}{\sqrt{n}} (x+3)^n$.

[Marks: 5]

Soln. $\lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{\sqrt{n+1}} (x+3)^{n+1} \times \frac{\sqrt{n}}{(-2)^n (x+3)^n} \right|$

$= 2|x+3|$

For abs. conv. $2|x+3| < 1 \Rightarrow |x+3| < \frac{1}{2}$

$\Leftrightarrow -\frac{1}{2} < x+3 < \frac{1}{2}$

$-\frac{7}{2} < x < -\frac{5}{2}$ 2

At $x = -\frac{7}{2}$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(-\frac{1}{2}\right)^n = \frac{1}{\sqrt{n}}$ which is a div. p-series. 1

At $x = -\frac{5}{2}$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$,

which is conv. by AST. 1

Hence Interval of conv.: $\left(-\frac{7}{2}, -\frac{5}{2}\right]$

Radius of conv.: $r = \frac{1}{2}$ 1

Q#4) Determine whether the series is absolutely conv., cond. conv., or divergent: $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln(n))}$

[Marks: 5]

Soln. By AST, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln(n))}$ is conv. 2

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ is div. by Integral test (TO verify the hypotheses of the Integral test)

$\lim_{t\to\infty} \int_2^t \frac{1}{x \ln(x)} dx = \lim_{t\to\infty} [\ln x]_2^t$

$= \lim_{t\to\infty} [\ln t - \ln 2]$

$= \infty$; Div. 2

Put $\ln x = u$,

$\frac{1}{x} dx = du$

$= \int \frac{1}{u} du = \ln(u)$

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$ is cond. conv. 1

Q #5) Find the Taylor Series for the function $f(x) = \cos x$ at $c = \pi$ [Marks: 5]

Soln. We have, the Taylor Series

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + f'''(c)\frac{(x-c)^3}{3!} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!} + \cdots \quad (1)$$

1

Here $f(x) = \cos x \Rightarrow f(\pi) = \cos\pi = -1$

$$f'(x) = -\sin x \Rightarrow f'(\pi) = 0$$

$$f''(x) = -\cos x \Rightarrow f''(\pi) = 1$$

$$f'''(x) = \sin x \Rightarrow f'''(\pi) = 0 \qquad 1$$

$$f^{(iv)}(x) = \cos x \Rightarrow f^{(iv)}(x) = -1$$

$\vdots$

Substituting these values in (1), we get

$$f(x) = \cos x = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \cdots + (-1)^{n+1}\frac{(x-\pi)^{2n}}{2n!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n!)} \qquad 3$$