Operations Analysis II

IE 322

Integer Programming

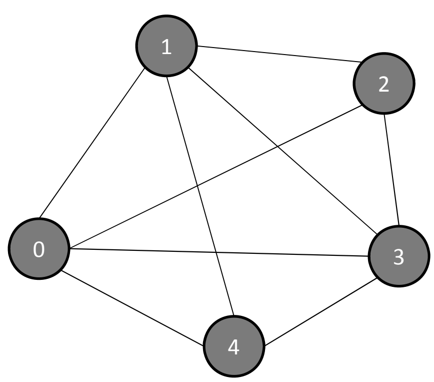
Definition

- Integer Linear Programs are linear programs with some or all the variables restricted to integer or (discrete) values.
- If all the variables restricted to integer or (discrete) values, then the Problem is an integer linear program ILP.
- If some variables are integer or (discrete) (but not all) then the problem is a Mixed Integer Linear Program MILP.

Challenge of Combinatorial Problems

Example: Travelling Salesman Problem TSP
Consider a traveling salesman who leaves a depot and return after delivering his product to a finite number of sales points. The distance between these points is known. His problem is to find the shortest route through all the sales points.

Travelling Salesman Problem



Node 0 represents the depot

Nodes 1, 2, 3 and 4 represent the sales points

	0	1	2	3	4
0	-	5	9	12	2
1	5	-	4	1	3
2	9	4	-	2	-
3	12	1	2	-	3
4	2	3	-	3	-

Matrix of distances (km)

Travelling Salesman Problem

Each sequence that begins with the depot, ends with the depot and includes all the sales points is a solution.

How many solutions exist? ⟹ $4! = 4\times3\times2=24$

All the solutions are feasible? ⟹ NO

$0 \xrightarrow{2} 4 \xrightarrow{3} 1 \xrightarrow{4} 2 \xrightarrow{2} 3 \xrightarrow{12} 0$ ⟹ Feasible solution

Distance = 23

$0 \longrightarrow 4 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1 \longrightarrow 0$ ⟹ Unfeasible solution

Impossible to go from 4 to 2

Travelling Salesman Problem

Heuristic

A heuristic Solution is a feasible solution based on an intelligent intuition . But without proof of the optimality

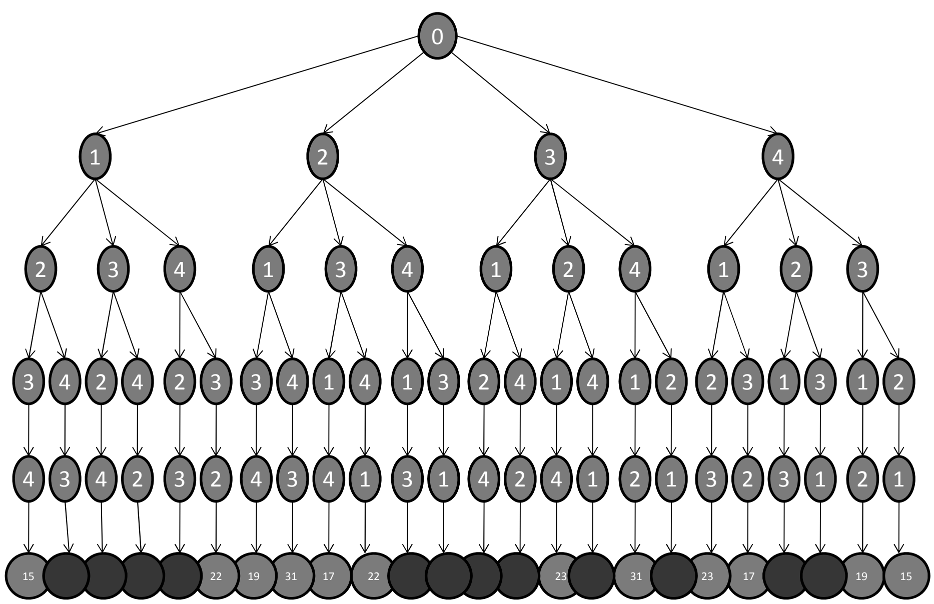
Example

Let ‘s move always to the nearest sales point ⟸ This is just an intuition

$0 \xrightarrow{2} 4 \xrightarrow{3} 1 \xrightarrow{1} 3 \xrightarrow{2} 2 \xrightarrow{9} 0$ ⟹ Distance = 17

We have a nice solution .But we don’t have a proof that it’s the optimal (best).

How to find the optimal Solution?



The manual computation of each sequence takes at least 1 minute

We have 24 sequences. So we need 24 minutes.

Suppose that the salesman has only one supplementary sales point

The number of solutions will be 5! = 5×4×3×2
= 5 × 4!
= 124 minutes ≥ 2 hours

We have added only one sales point. But the computation time is multiplied by 5.

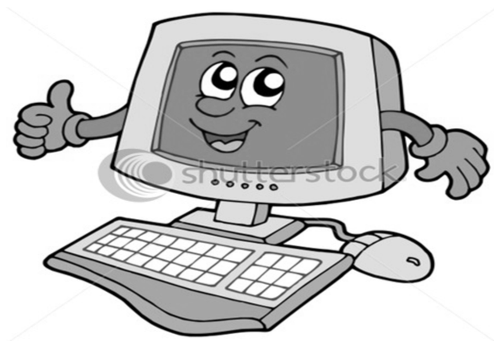
The number of sequences explodes according to the number of sales points

If the number of sales points is 10 then the computation time = 10! minutes
= 3749760 minutes
= 62492 hours
= 2604 days **≥ 7 years**

How can I be faster to find the optimal Solution?

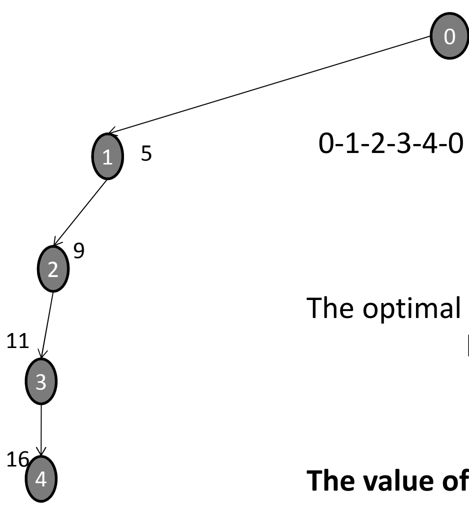
A strong computer can take just 0.000001 second to check the feasibility of a sequence or to compute its distance.

We have 10 nodes. So we have 10! sequences. The computation time of the computer is = 10! ×0.000001 = 3749760 ×0.000001 = 3.7 seconds.



If the number of sales points is 20 then the computation time of the computer
= 20!x0.000001 seconds
= 2432902008176640000 x0.000001 seconds
= 2432902008177 seconds
= 675806113 hours
= 28158588 days

≥ 78218 years



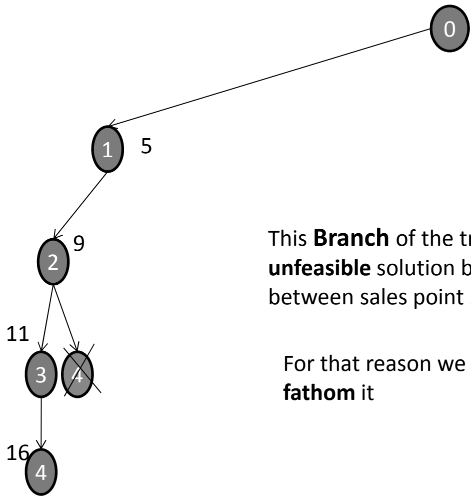
0-1-2-3-4-0 is a feasible solution with distance 16

The optimal solution will be equal or less then 16

The value of any feasible solution is an Upper Bound of the value of the optimal solution in case of minimization problems

Upper Bound = 16

UB = 16

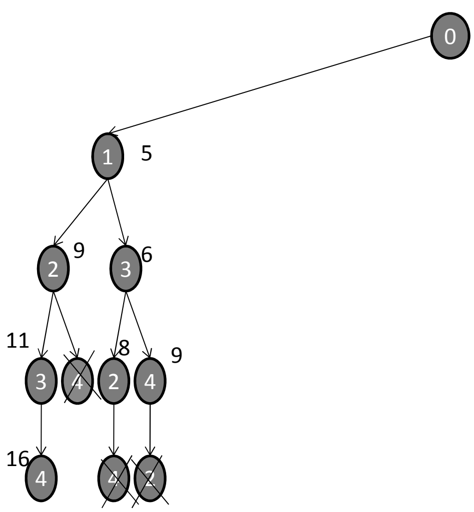


This **Branch** of the tree search will lead always to an **unfeasible** solution because there is no direct path between sales point 2 and sales point 4

For that reason we don't explore it and we **fathom** it

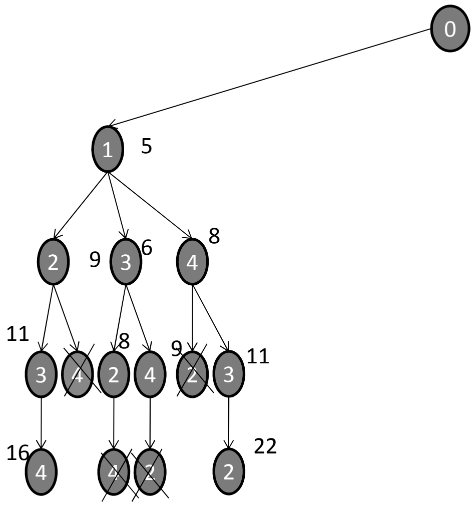
Upper Bound = 16

UB = 16



Upper Bound = 16

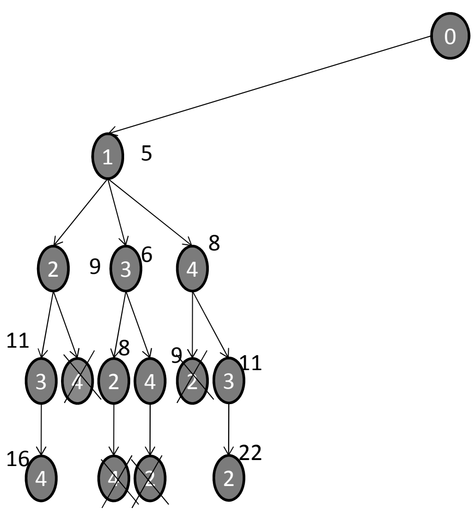
UB = 16



This is a second feasible solution

Upper Bound = 16

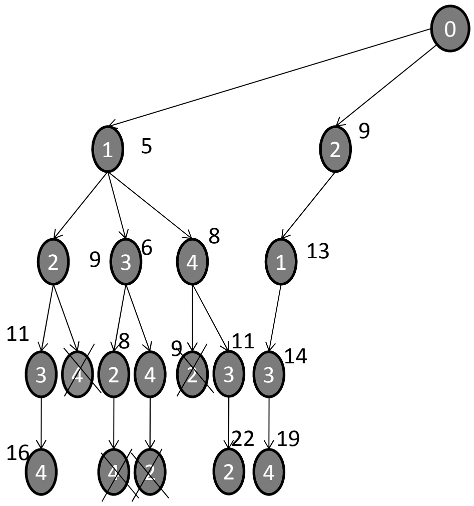
UB = 16



This is a second feasible solution

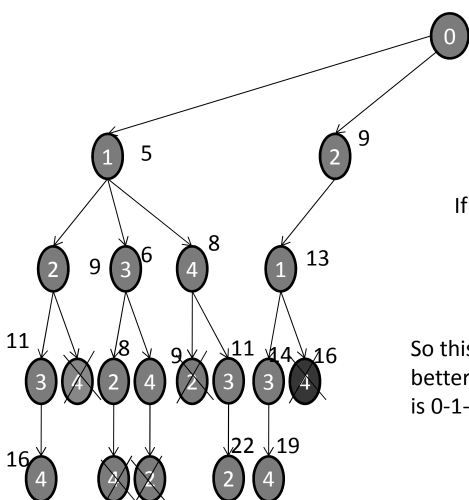
Upper Bound = 16

UB = 16



Upper Bound = 16

UB = 16



If we take the route 0-2-1-4 we will travel at least 16 km

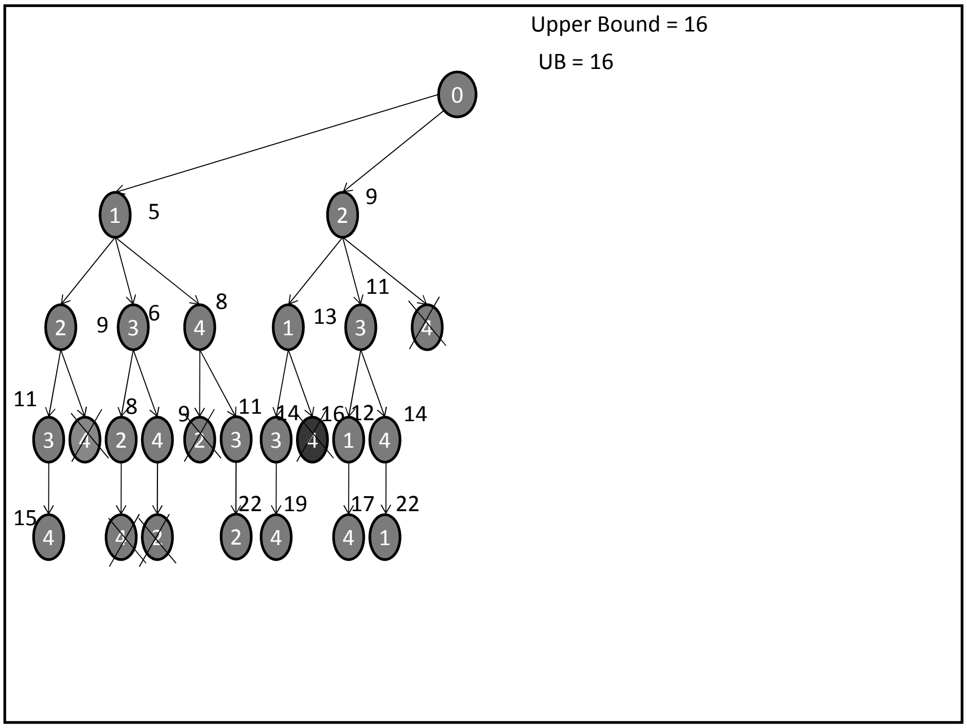
If we continue in this route we will travel more than 16 km

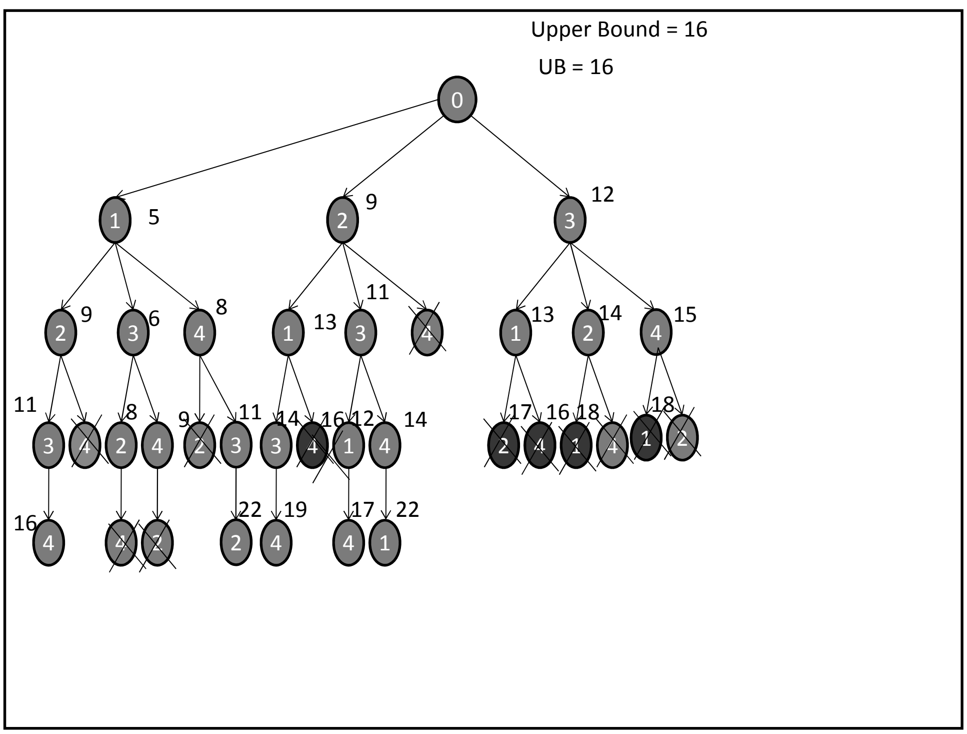
So this route will never lead to a solution better than the best existing solution which is 0-1-2-3-4-0 with 16 km as a distance

It's better to fathom this branch

16 is the Lower Bound (LB) of the read node

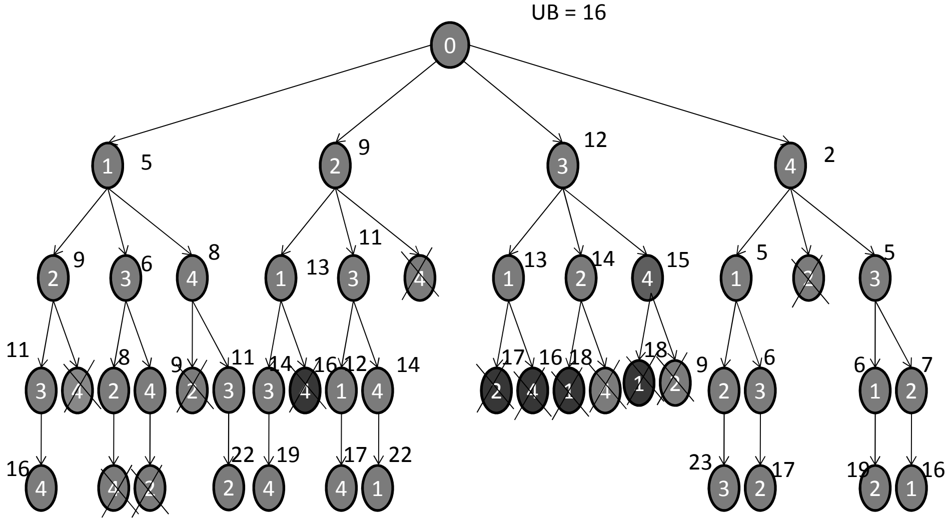
[redacted]


Upper Bound = 16
UB = 16


Upper Bound = 16
UB = 16

Upper Bound = 16

UB = 16



The Shortest route is 16 km

What we have done in the previews intelligent enumeration is a:

Branch And Bound Algorithm

Objective of Integer Linear Programming

Objective of Integer Linear Programming is modeling combinatorial problems as integer programs and solve theme with specific methods as :

- Branch and Bound.
- Branch and Price.
- Branch and Cut.
- Cutting plan methods.
- Decomposition methods.

When you study ILP, you need to concentrate on three areas :

- Applications
- Theory
- Computation

Applications

- Cutting and Packing Problems.
- Network Design and Transportations Problems.
- Scheduling and Time Tabling Problems.
- Location, Assignment and Supply chain Problems.
- Various applications in real life problems.

Example 9-1-1 (Project Selection)

Five projects are being evaluated over a 3-year planning table gives the expected returns for each project and the associated yearly expenditures

	expenditures (million $)/yr			
projects	**1**	**2**	**3**	**Returns (million $)**
1	**5**	**1**	**8**	**20**
2	**4**	**7**	**10**	**40**
3	**3**	**9**	**2**	**20**
4	**7**	**4**	**1**	**15**
5	**8**	**6**	**10**	**30**
Available funds (million $)	**25**	**25**	**25**	

Which project should be selected over the 3-year horizon?

Problem Set 9.1A

Problem (1)

Modify and solve the project selection problem to account for the following additional restrictions :

a. Project 5 must be selected if either project 1 or project 3 is selected.
b. Projects 2 and 3 are mutually exclusive.

Problem (2)

Five items are to be loaded in a vessel. The weight w_j , volume v_j and the value r_j for item j are given in the following table. The maximum allowable cargo weight and volume are 111 tons and 109 m^3.

item j	Unit weight wj	Unit Volume vj	Unit Value rj
1	5	1	4
2	8	8	7
3	3	6	6
4	2	5	5
5	7	4	4

Formulate the ILP model of the most valuable cargo problem.

Problem 7

You have the following three-letter words : AET, FAR, TVA, ADV, JOE, FIN, OSF and KEN. Suppose that we assign a numeric values to the alphabet starting with A = 1 and ending with Z = 26. Each word is sorted by adding numeric codes of its three letters. For example, AFT has a score of 1+6+20 =27. You are to select five of the given eight words that yield the maximum total score. Simultaneously, the selected five words must satisfy the following conditions :
Sum of letter 1 scores < Sum of letter 2 scores < Sum of Letter 3 scores

Formulate the Problem as an ILP and find the optimum solution.

Problem 8

Solve Problem 7 given that, in addition to the total sum being the largest, the sum of column 1 and column 2 will be the largest as well.

Problem 10

The Record-a-Song company has contracted with a rising star to record eight songs. The duration of the different songs are 8, 3, 5, 5, 9, 6, 7 and 12 minutes, respectively. Record-a-Song uses a two-sided cassette tape for the recording . Each side has a capacity of 30 minutes. The company would like to distribute the songs between the two sides such that the length of the songs on each side is about the same. Formulate the problem as an ILP.

Problem 11

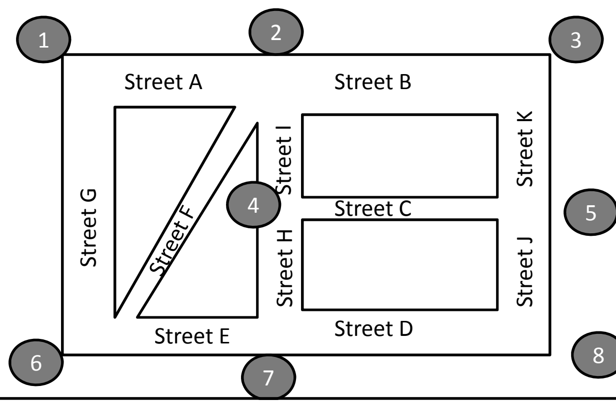
In Problem 10 suppose that the nature of the melodies dictates that songs 3 and 4 cannot be recorded on the same side. Formulate the problem as an ILP.
Would it be possible to use a 25-minutes tape (each side) to record the eight songs?
If not use ILP to determine the minimum tape capacity needed to make the recording.

Example 9-1-1 (Installing security telephones)

To promote on-campus safety, the security department is in process of installing emergency telephones at selected locations. The department wants to install the minimum number of telephones, provided that each of the main street of the car is served by at least one telephone. The following figure maps the principal street to K) on the campus.

It's logical to place the telephones at street intersections so that each telephone serve at least two streets.

Give the ILP that minimize the number of telephones to install.



Problem Set 9.1B

Problem 1

ABC is an ILT trucking company that delivers loads to five customers . The following list provides the customers associated with each route :

Route	Customers served on the route
1	1,2,3,4
2	4,3,5
3	1,2,5
4	2,3,5
5	1,4,2
6	1,3,5

The following matrix lists distances among the truck terminal and the customers

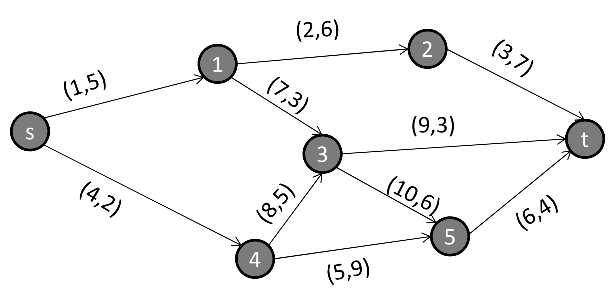
	ABC	1	2	3	4	5
ABC	0	10	12	16	9	8
1	10	0	32	8	17	10
2	12	32	0	14	21	20
3	16	8	14	0	15	18
4	9	17	21	15	0	11
5	8	10	20	18	11	0

The objective is to determine the least distance needed to make the daily deliveries to all five customers

Network Design Problems

Shortest path Problem

Consider the graph G (V, E) where V is the set of nodes and E is the set of arcs. Each arc e E has a distance c_e . Formulate the shortest path problem from node s to node t as a linear program.



(e, c_e) = (number of the arc, cost of the arc)

$$x_e = \begin{cases} 1 \text{ if arc e is considerd in the path} \\ 0 \text{ otherwise} \end{cases}$$

Objective function

$$Min \sum_{e \in E} c_e x_e$$

Constraints

$$x_1 + x_4 = 1$$
$$x_1 - x_2 - x_7 = 0$$
$$x_2 - x_3 = 0$$
$$x_7 + x_8 - x_9 - x_{10} = 0$$
$$x_4 - x_5 - x_8 = 0$$
$$x_5 + x_{10} - x_6 = 0$$
$$x_3 + x_6 + x_9 = 1$$
$$x_e \in \{0,1\} \ \forall e \in E$$

$$\text{let } \delta^+(i) = \{\text{set of arcs leaving node i}\}$$
$$\text{let } \delta^-(i) = \{\text{set of arcs insident to node i}\}$$

the formulation is

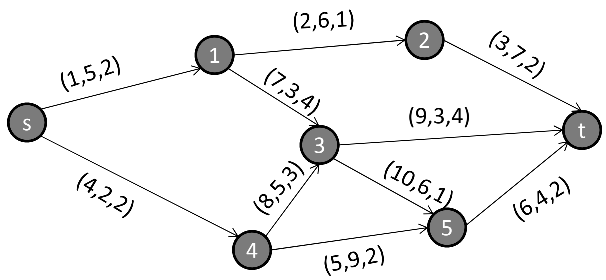
$$\text{Min} \sum_{e \in E} c_e x_e$$

$S.T$

$$\sum_{e \in \delta^+(s)} x_e = 1$$
$$\sum_{e \in \delta^+(i)} x_e - \sum_{e \in \delta^-(i)} x_e = 0 \ \forall i \in V / \{s,t\}$$
$$\sum_{e \in \delta^-(t)} x_e = 1$$
$$x_e \in \{0,1\} \ \forall e \in E$$

Min cost flow

Consider the graph G (V, E) where V is the set of nodes and E is the set of arcs. Each arc e E has a capacity c_e and a unit cost of flow α_e . We need to send a quantity of flow d=6 from node s to t with minimum total cost. Formulate the problem as a linear program



$(e, c_{e,} \alpha_e)$ = (number of the arc, capacity of the arc, unit cost of flow on the arc)

x_e = the total amount of flow that passes on arc e

Objective function

$$Min \sum_{e \in E} \alpha_e x_e$$

Constraints

$$x_1 + x_4 = 6$$
$$x_1 - x_2 - x_7 = 0$$
$$x_2 - x_3 = 0$$
$$x_7 + x_8 - x_9 - x_{10} = 0$$
$$x_4 - x_5 - x_8 = 0$$
$$x_5 + x_{10} - x_6 = 0$$
$$x_3 + x_6 + x_9 = 6$$
$$x_e \le c_e \quad \forall e \in E$$
$$x_e \ge 0 \quad \forall e \in E$$

let $\delta^{+}(i) = \{\text{set of arcs leaving node i}\}$

let $\delta^{-}(i) = \{\text{set of arcs going to node i}\}$

the formulatio n is

$$\text{Min} \sum_{e \in E} \delta_e x_e$$

$S.T$

$$\sum_{e \in \delta^{-}(s)} x_e = d$$

$$\sum_{e \in \delta^{-}(i)} x_e - \sum_{e \in \delta^{-}(i)} x_e = 0 \ \forall i \in V / \{s, t\}$$

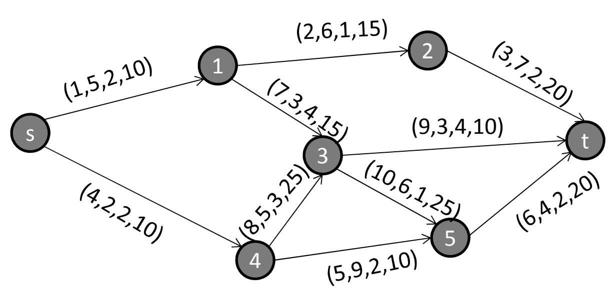
$$\sum_{e \in \delta^{-}(t)} x_e = d$$

$$x_e \le c_e \quad \forall e \in E$$

$$x_e \ge 0 \quad \forall e \in E$$

Min cost flow with fixed charge

Consider the graph G (V, E) where V is the set of nodes and E is the set of arcs. Each arc e □ E has a capacity c_e ,a unit cost of flow α_e and a fixed cost f_e of installing the arc e. We need to send a quantity of flow d=6 from node s to t with minimum total cost. Formulate the problem as a linear program



(e, $c_{e,}$ $\alpha_{e\,,}$ f_e) = (number of the arc, capacity of the arc, unit cost of flow on the arc, fixed cost of installing the arc)

x_e = the total amount of flow that passes on arc e

$$y_e = \begin{cases} 1 \text{ if arc e is used} \\ 0 \text{ otherwise} \end{cases}$$

Objective function

$$Min \sum_{e \in E} \alpha_e x_e + \sum_{e \in E} f_e y_e$$

Constraints

$$x_1 + x_4 = 6$$
$$x_1 - x_2 - x_7 = 0$$
$$x_2 - x_3 = 0$$
$$x_7 + x_8 - x_9 - x_{10} = 0$$
$$x_4 - x_5 - x_8 = 0$$
$$x_5 + x_{10} - x_6 = 0$$
$$x_3 + x_6 + x_9 = 6$$
$$x_e \le c_e y_e \ \forall e \in E$$
$$x_e \ge 0 \quad \forall e \in E$$
$$y_e \in \{0,1\} \forall e \in E$$

let $\delta^+(i) = \{\text{set of arcs leaving node i}\}$

let $\delta^-(i) = \{\text{set of arcs going to node i}\}$

the formulatio n is

$$Min \sum_{e \in E} \alpha_e x_e + \sum_{e \in E} f_e y_e$$

$S.T$

$$\sum_{e \in \delta^-(s)} x_e = d$$

$$\sum_{e \in \delta^+(i)} x_e - \sum_{e \in \delta^-(i)} x_e = 0 \ \forall i \in V / \{s,t\}$$

$$\sum_{e \in \delta^-(t)} x_e = d$$

$$x_e \le c_e y_e \ \forall e \in E$$
$$x_e \ge 0 \quad \forall e \in E$$
$$y_e \in \{0,1\} \forall e \in E$$

Job-Sequencing Model

Example 9.1-4

Jobco uses a single machine to process three jobs. (Both the processing time and the due date (in days) for each job are given in the following tables. The due dates are measured from zero, the assumed start time of the first job.

job (j)	proceesing time (pj)	Due date (dj)	Late penality (lj)
1	5	25	19
2	20	22	12
3	15	35	34

The objective of the problem is to determine the minimum late penalty sequence for processing the three jobs

Example 9.1.D

Jobco Shop has 7 jobs to be processed on a single machine. (Both the processing time and the due date (in days) for each job are given in the following tables).

job (j)	proceesing time (pj)	Due date (dj)
1	10	40
2	3	20
3	13	30
4	15	30
5	9	45
6	5	15
7	8	10

If job 4 precedes job 3 then job 2 must precede job 6. The objective is to minimize the total late.

Integer Programming Algorithms

B&B and Cutting Plane Methods

The two commonly used methods are:

1. Branch and bound method
2. Cutting Plane Method

Neither method is consistently effective; but B&B is far more successful.

Branch-and-Bound (B&B)

- Developed in 1960 by A Land and G Doig
- Relax the integer restrictions in the problem and solve it as a regular LP. Let's call this LP_1.
- Test if integer requirements are met. Else branch to get sub-problems LP_2 and LP_3.

Branching

- If LP1 (in general LP_i) fails to yield integer solution, branch on any variable that fails to meet this requirement.

  Note: In mixed integer problems, a continuous variable is never selected for branching.

Bounding / Fathoming

- Select LP_1 (in general LP_i) and solve. Three conditions arise.
  - Infeasible solution, declare fathomed (no further investigation of LP_i).
  - Integer solution. If it is better than the current best solution update the current best. Declare fathomed.
  - Non-integer solution. If it is worse than the current best, declare fathomed. Else branch again.

Best Bound

- In maximisation, the solution to a sub-problem is superior if it's integer and raises the current lower bound.
- In minimisation, the solution to a sub-problem is superior if it's integer and lowers the current upper bound.
- When all sub-problems have been fathomed, stop. The current bound is the best bound.

Branch and Bound Example

Maximise $z=5\,x_1 + 4x_2$

Subject to:

$x_1 + \ x_2 \leq 5$

$10x_1 + 6x_2 \leq 45$

$x_1, x_2 \geq 0$ and integer

Relax ILP $\Rightarrow$ Remove the integrality constraint $\Rightarrow$ Get LP1

Maximise $z=5\,x_1 + 4x_2$

Subject to:

$x_1 + \ x_2 \leq 5$

$10x_1 + 6x_2 \leq 45$

$x_1, x_2 \geq 0$

LP1

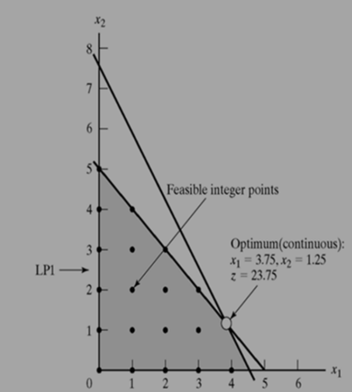


Figure 9.6
Solution spaces for ILP (lattice points) and LP1 (shaded area) of Example 9.2-1.

Branch on x_1

LP3

Maximise $z=5\,x_1+ 4x_2$
Subject to:
$x_1 + x_2 \le 5$
$10x_1 + 6x_2 \le 45$
$x_1 \le 3$
$x_1, x_2 \ge 0$

LP2

Maximise $z=5\,x_1+ 4x_2$
Subject to:
$x_1 + x_2 \le 5$
$10x_1 + 6x_2 \le 45$
$x_1 \ge 4$
$x_1, x_2 \ge 0$

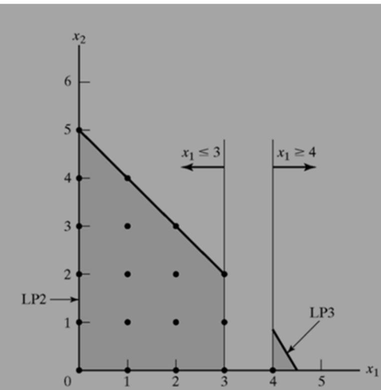


Figure 9.7
Solution Spaces of LP2 and LP3 for Example 9.2-1.

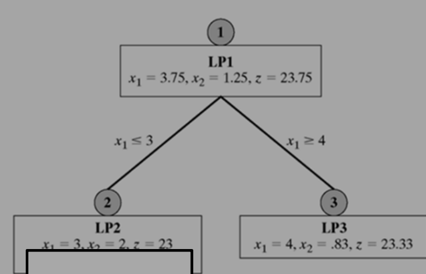


Figure 9.8
Using branching variable x_1 to create LP2 and LP3 for Example 9.2-1.

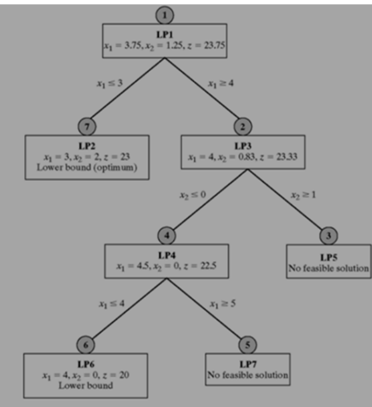
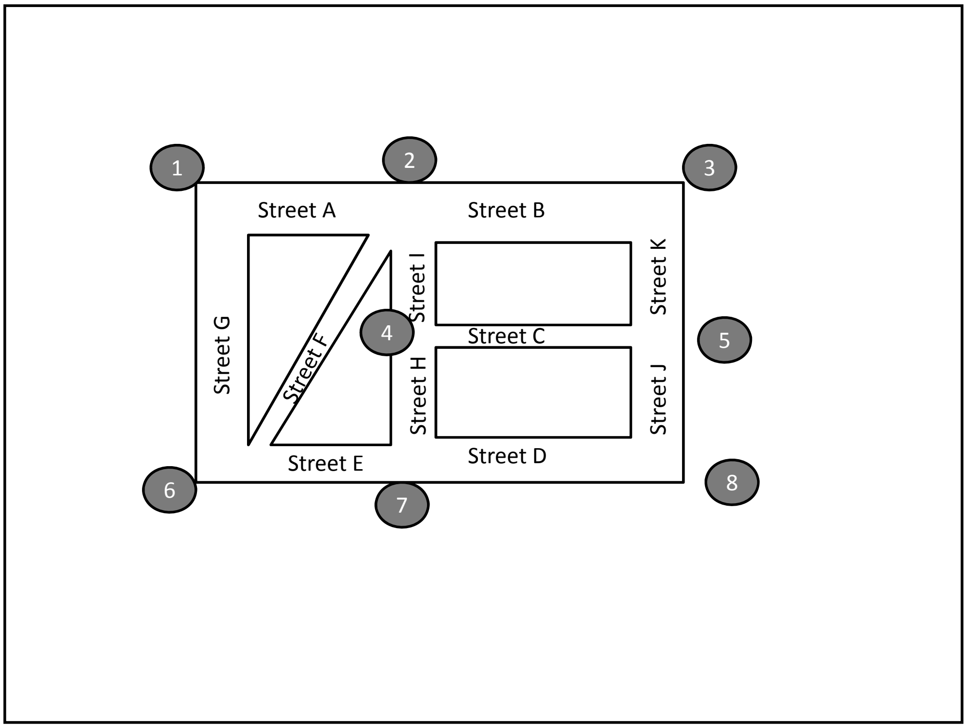


Figure 9.9
Alternative B&B tree for Example 9.2-1.

Home Work

- Solve Problems (9.1C.1, 9.1C.2).
- Solve Problems (9.1D.4, 9.1D.9).
- Solve Problems (9.2A.2).


1
2
3
Street A
Street B
Street I
Street K
4
5
Street G
Street F
Street C
Street H
Street J
Street E
Street D
6
7
8