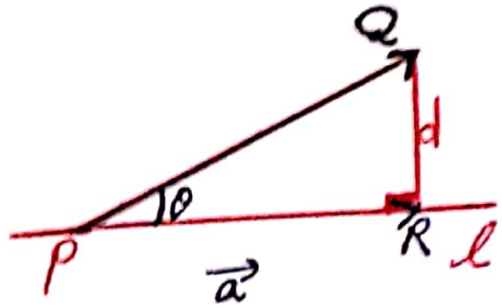
Vectors & line and plane

Lecture 20

Defn ①

* The distance between a point Q and ℓ is



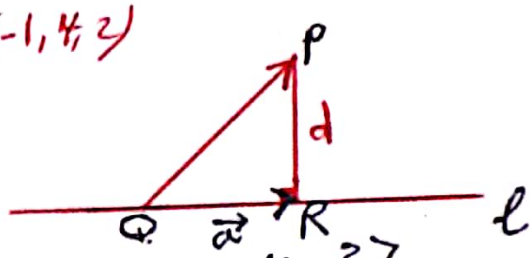
$$d = \|\overrightarrow{PQ}\| \sin\theta$$

$$d = \frac{\|\overrightarrow{PQ}\|\,\|\overrightarrow{PR}\| \sin\theta}{\|\overrightarrow{PR}\|}$$

$$d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PR}\|} = \frac{\|\overrightarrow{PQ} \times \vec{a}\|}{\|\vec{a}\|}$$

where $\vec{a}$ is the direction vector for line ℓ and P is a given point on ℓ.

Ex Find the distance from $P(3,1,-2)$ to the line ℓ which passes through $Q(2,5,1)$ and $R(-1,4,2)$



Ans: $d = \dfrac{\|\overrightarrow{QP} \times \vec{a}\|}{\|\vec{a}\|}$

$$\overrightarrow{QP} = \langle 3-2,\, 1-5,\, -2-1 \rangle = \langle 1, -4, -3 \rangle$$

$$\vec{a} = \overrightarrow{QR} = \langle -1, 4, 2 \rangle - \langle 2, 5, 1 \rangle$$

$$\vec{a} = \langle -3, -1, 1 \rangle$$

$$\overrightarrow{QP} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & -3 \\ -3 & -1 & 1 \end{vmatrix} = -7\vec{i} + 8\vec{j} - 13\vec{k}$$

$$\|\overrightarrow{QP} \times \vec{a}\| = \sqrt{49 + 64 + 169} = \sqrt{282} = 16.79$$

$$\|\vec{a}\| = \sqrt{9+1+1} = \sqrt{11} = 3.3$$

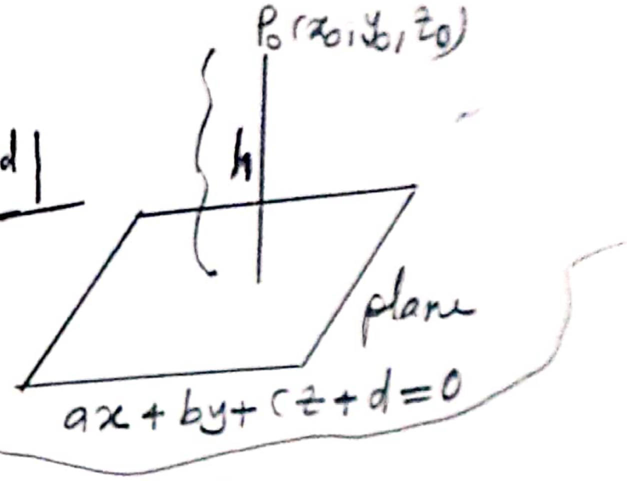
$$\therefore d = \frac{16.79}{3.3} \simeq 5.09 \text{ unit length.}$$

Def$_n$ ②

Distance from a point $P_0(x_0, y_0, z_0)$ to the plane

$ax + by + cz + d = 0$

is
$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$



EX ② Find the distance from $P(1,-1,2)$ to the plane $3x-7y+z-5=0$

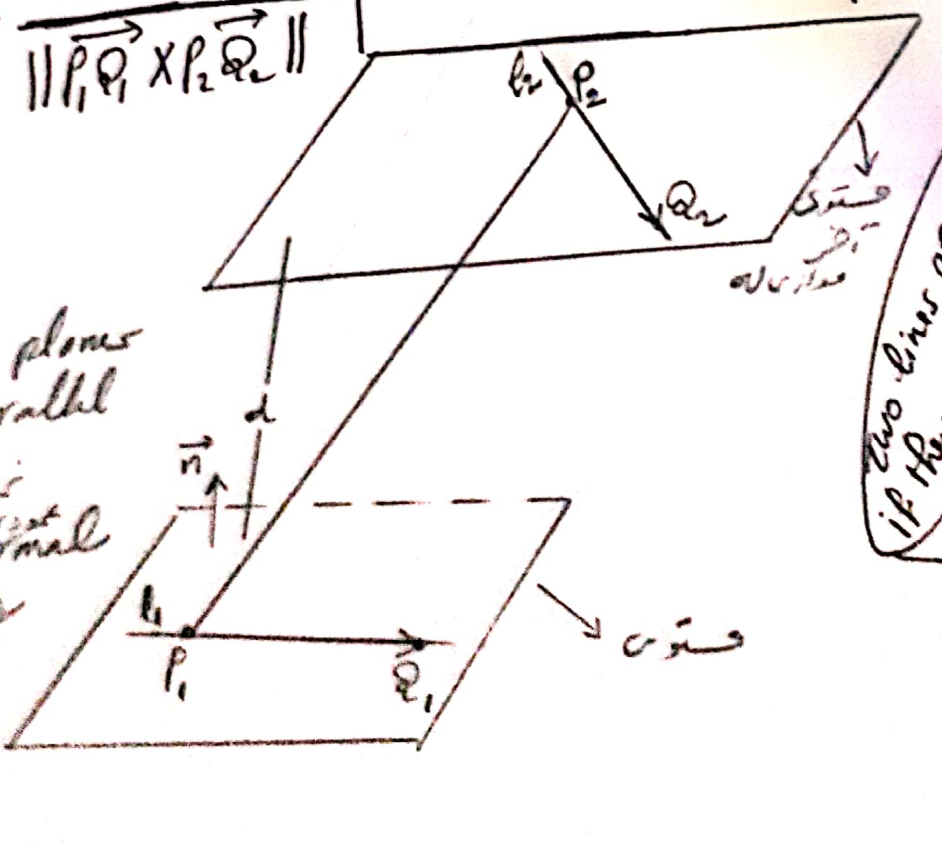
Ans:
$$h = \frac{|3(1)-7(-1)+2-5|}{\sqrt{9+49+1}} = \frac{7}{\sqrt{59}}$$

$h \simeq 0.91$ unit length.

Def$_n$ ③

* Shortest distance d between skew lines l_1 and l_2 is

$$d = \frac{1}{\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\|} \left| (\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}) \cdot \overrightarrow{P_1P_2} \right|$$



Note

The two planes are parallel and $\vec{n}$ is a unit normal vector for both of them.

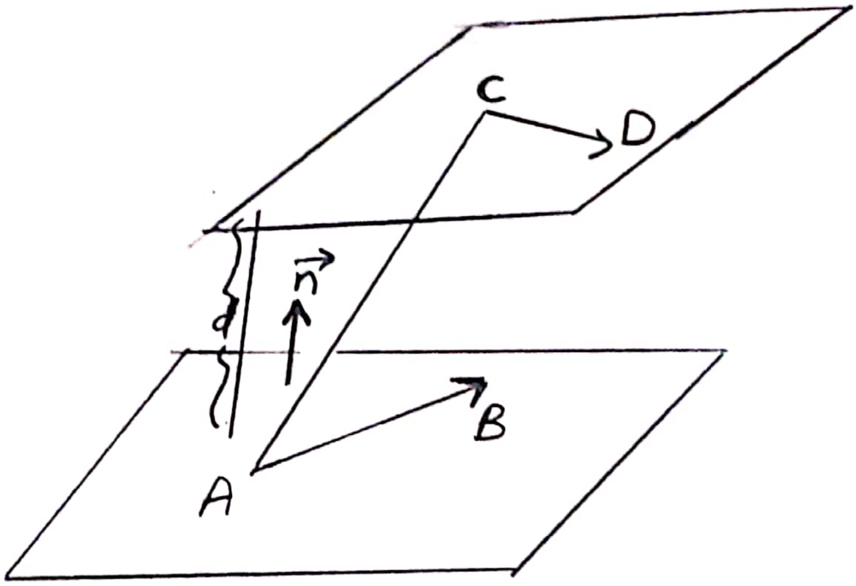
Note

two lines are called skew if they are not parallel and donot intersect.

EX 3

Find the shortest distance between skew lines, line l_1 through points $A(1,2,4)$ and $B(3,2,5)$ and line l_2 through points $C(6,3,3)$ and $D(4,5,1)$

Ans:



$\vec{AB} = \langle 2,0,1 \rangle$, $\vec{CD} = \langle -2,2,-2 \rangle$, $\vec{AC} = \langle 5,1,-1 \rangle$

the shortest distance is

$$d = \frac{1}{\|\vec{AB} \times \vec{CD}\|} \left| (\vec{AB} \times \vec{CD}) \cdot \vec{AC} \right|$$

$$\vec{AB} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ -2 & 2 & -2 \end{vmatrix} = -2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\|AB \times \vec{CD}\| = \sqrt{4+4+16} = \sqrt{24}$$

$$\vec{AB} \times \vec{CD} \cdot \vec{AC} = \langle -2,2,4 \rangle \cdot \langle 5,1,-1 \rangle$$

$$= -10 + 2 - 4 = -12$$

$$\therefore d = \frac{12}{\sqrt{24}} = \frac{12}{2\sqrt{6}} = \frac{6}{\sqrt{6}} = \sqrt{6} \text{ unit length.}$$

4/ Note that

$$\left|(\overrightarrow{AB} \times \overrightarrow{CD}) \cdot \overrightarrow{AC}\right| = \left|\underbrace{\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CD})}_{\text{triple scalar product}}\right|$$

$$\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CD}) = \begin{vmatrix} 5 & 1 & -1 \\ 2 & 0 & 1 \\ -2 & 2 & -2 \end{vmatrix} = -12$$

$$\Rightarrow d = \frac{12}{\sqrt{24}} = \sqrt{6} \text{ unit length.}$$