Vector Valued Functions

- **Def^n**

$\vec{r}$ is a vector valued function iff

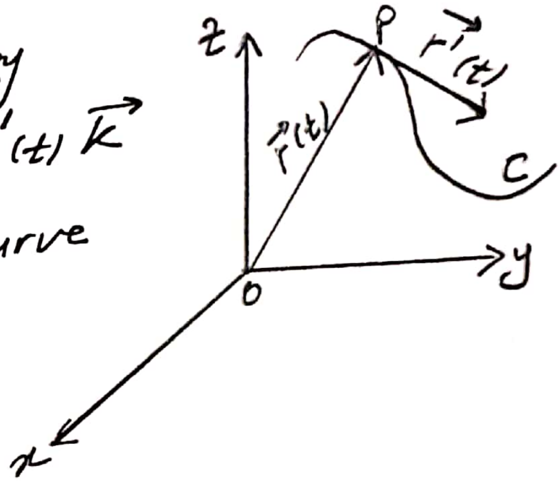
$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

where $t \in \mathbb{R}$ and $\vec{r}(t) \in \mathbb{R}^3$ space

* The derivative of $\vec{r}$ is $\vec{r}'$ defined by

$$\vec{r}'(t) = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$$

which is a tangent vector to the curve C at point P as in opp. fig.



- **Applications: Motion in space**

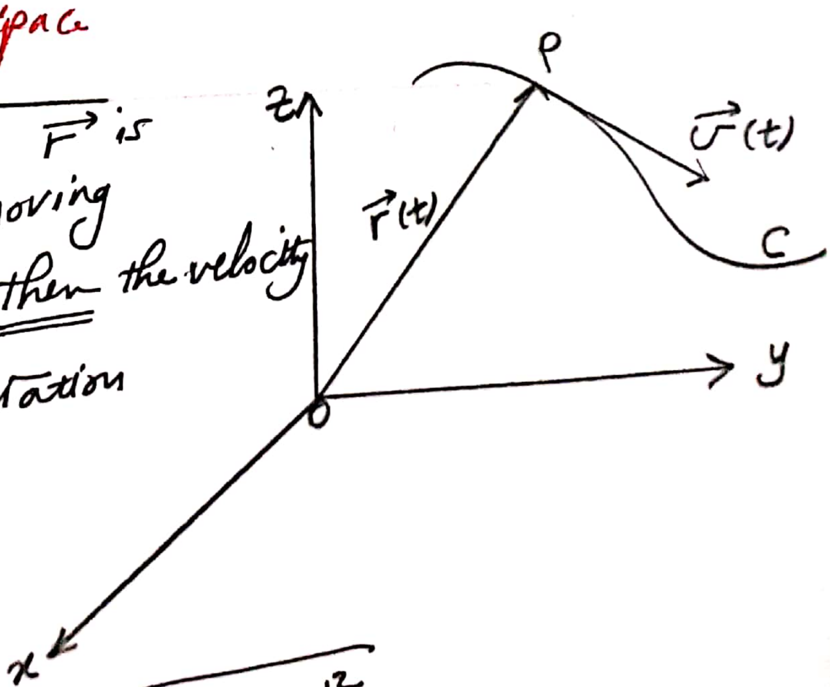
If the vector valued fn $\vec{r}$ is a position vector of a moving particle P at time t then the velocity

$$\vec{v} = \frac{d\vec{r}}{dt},$$

the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

and the speed is



$$\|\vec{v}\| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \quad m/sec,\ ft/sec, \ldots$$

Ex ① Find velocity, speed and acceleration of the vector valued function of the particle moving along the curve

$$\vec{r}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j} + t^2\,\vec{k}, \quad t = \frac{\pi}{2}$$

Ans:

$$\vec{r}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j} + t^2\,\vec{k}$$

velocity $\vec{v}(t) = \vec{r}\,'(t)$

$$\vec{v}(t) = (-t\sin t + \cos t)\,\vec{i} + (t\cos t + \sin t)\,\vec{j} + 2t\,\vec{k}$$

$$\vec{v}\Big|_{t=\frac{\pi}{2}} = -\frac{\pi}{2}\,\vec{i} + \vec{j} + \pi\,\vec{k}$$

$$\text{speed} = \sqrt{\frac{\pi^2}{4} + 1 + \pi^2} = \sqrt{\frac{5\pi^2}{4} + 1}$$

and acceleration $\vec{a}(t) = \vec{r}\,''(t)$

$$\vec{a}(t) = (-t\cos t - 2\sin t)\,\vec{i} + (2\cos t - t\sin t)\,\vec{j} + 2\vec{k}$$

$$\vec{a}\Big|_{t=\frac{\pi}{2}} = -2\vec{i} - \frac{\pi}{2}\,\vec{j} + 2\vec{k}$$

- **Integration of Vector Valued Function**

$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \vec{v}(t) = \int \vec{a}\,dt + c_1$$

and

$$\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow \vec{r}(t) = \int \vec{v}\,dt + c_2$$

Ex ② Find the path of the curve when acceleration of the particle moving along the curve is $\vec{a}(t) = -2\cos t\,\vec{i} - 2\sin t\,\vec{j} + 2\vec{k}$, initial velocity of the particle is $\vec{v}(0) = 2\vec{j}$ and it starts from the point $(2,0,0)$

Ans:

$$\vec{a}(t) = -2\cos t\,\vec{i} - 2\sin t\,\vec{j} + 2\vec{k}$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$\vec{v}(t) = \int [-2\cos t\,\vec{i} - 2\sin t\,\vec{j} + 2\vec{k}]\,dt$$

$$\vec{v}(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 2t\,\vec{k} + C_1$$

at $t=0$, $\vec{v}(0) = 2\vec{j}$

$$\Rightarrow 2\vec{j} = \cancel{-2\sin 0\,\vec{i}} + 2\cos 0\,\vec{j} + \cancel{2(0)\vec{k}} + C_1$$

$$2\vec{j} = 2\vec{j} + C_1 \qquad \therefore C_1 = \vec{0}$$

$$\therefore \boxed{\vec{v}(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 2t\,\vec{k}}$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int [-2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 2t\,\vec{k}]\,dt$$

$$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t^2\vec{k} + C_2$$

$$\vec{r}(0) = 2\vec{i} + \cancel{2\sin 0\,\vec{j}} + \cancel{0^2\vec{k}} + C_2$$

$\because$ the particle starts from $(2,0,0) \Rightarrow \vec{r}(0) = 2\vec{i}$

$$\therefore 2\vec{i} = 2\vec{i} + C_2 \qquad \therefore C_2 = \vec{0}$$

$$\therefore \vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t^2\vec{k}$$

which determines the path of the curve. #

• Note that: $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$

$\int \cos x\,dx = \sin x + C$, $\int \sin x\,dx = -\cos x + C$

Ex ③

Sketch the path of an object moving along the plane curve given by the position vector

$$\vec{r}(t) = (t^2 - 4)\,\vec{i} + t\,\vec{j}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$

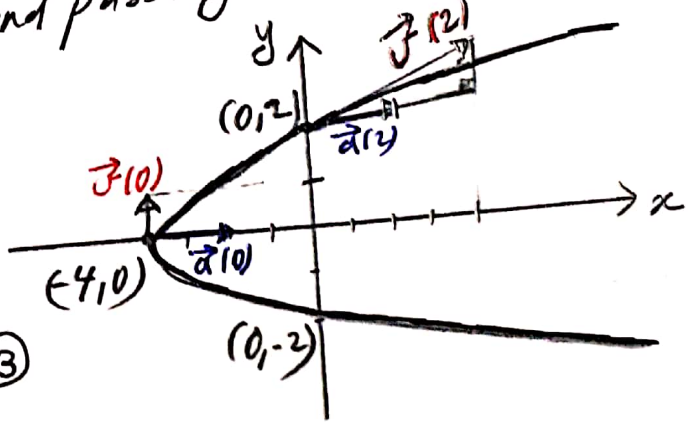
Ans:

$\because \vec{r}(t) = (t^2 - 4)\vec{i} + t\vec{j}$ ① $\Rightarrow$ at $t = 0$, $x = -4$, $y = 0$

$\therefore x = t^2 - 4$, $y = t$ $\Rightarrow$ the object starts its motion from the vertex $(-4, 0)$

$\therefore \boxed{x = y^2 - 4}$ ② which is a parabola

its vertex is $(-4, 0)$ and passing through $(0, 2)$, $(0, -2)$



③

$\because \vec{v}(t) = \dfrac{d\vec{r}}{dt}$

$\vec{v}(t) = 2t\,\vec{i} + \vec{j}$

$\vec{v}\big|_{t=0} = \vec{j}$ at $(-4, 0)$

$\vec{v}\big|_{t=2} = 4\vec{i} + \vec{j}$ at $(0, 2)$

④

$, \vec{a}(t) = \dfrac{d\vec{v}}{dt}$

$\vec{a}(t) = 2\vec{i}$

$\vec{a}\big|_{t=0} = 2\vec{i}$ at $(-4, 0)$

$\vec{a}\big|_{t=2} = 2\vec{i}$ at $(0, 2)$

Note that

The particle moving with a constant acceleration (Uniform acceleration) in right direction and its speed increasing as it moves away from the vertex of the parabola.