Partial Derivatives

Lecture (28)

Ex If $w = 3x^2 - xy$ find dw

where $(x,y): (1,2) \Rightarrow (1.01, 1.98)$

Ans: $dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$

$dw = (6x - y)\, dx - x\, dy$

at $x = 1$, $y = 2$, $dx \simeq \Delta x = 0.01$, $dy \simeq \Delta y = -0.02$

$dw = (6-2)(0.01) - 1(-0.02)$

$\therefore dw = 4(0.01) + 0.02 = 0.06$

Note If $w = f(x,y,z,t)$ the total differential

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial t} dt$$

Ex Find the total differential of fn

$w = x^2 z + 4yt^3 - xz^2 t$

Ans:

$\because dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial t} dt$

$\therefore dw = (2xz - z^2 t)dx + 4t^3 dy + (x^2 - 2xzt)dz$
$+ (12yt^2 - xz^2)dt$

• Chain Rule

If $w = f(u,v)$ be a function of two variables

, $u = u(x,y)$, $v = v(x,y)$ then

$$\left[\begin{array}{l} \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \\[2ex] \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} \end{array}\right.$$

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Note If $w = f(x,y,z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$

then $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

(By using chain Rule)

Ex Find $\frac{dw}{dt}$ if $w = x^2+y^2+z^2$, $x = t\cos t$, $y = t\sin t$, $z = t$

Ans: $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = 2x(\cos t - t\sin t) + 2y(\sin t + t\cos t) + 2z$$

Ex Find $\frac{\partial w}{\partial z}$ if $w = r^2 + sv + t^3$ with $r = x^2+y^2+z^2$, $s = xyz$, $v = xe^y$, $t = yz^2$

Ans: clearly, $w = w(r,s,v,t)$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$$

$$= 2r\cdot 2z + v\cdot xy + \cancel{s(0)} + 3t^2(2yz)$$

$$= 4rz + xyv + 6t^2yz$$

$$\therefore \frac{\partial w}{\partial z} = 4z(x^2+y^2+z^2) + x^2ye^y + 6y^3z^5$$

x, y, z $\frac{\partial w}{\partial z}$

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Ex If $w = f(x^2+y^2)$, show that $y\left(\frac{\partial w}{\partial x}\right) - x\left(\frac{\partial w}{\partial y}\right) = 0$

Ans: let $w = f(u)$, $u = x^2+y^2$

$$\therefore \frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial w}{\partial x} = \frac{dw}{du}(2x), \quad \frac{\partial w}{\partial y} = \frac{dw}{du}(2y)$$

$$\therefore y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} = \cancel{2xy\left(\frac{dw}{du}\right)} - \cancel{2xy\left(\frac{dw}{du}\right)} = 0$$

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Ex If $w = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow x = x(r,\theta)$ $\Rightarrow y = y(r,\theta)$

show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

Ans:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + 2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\cdot\cos\theta\sin\theta \quad (1)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 r^2\sin^2\theta + \left(\frac{\partial w}{\partial y}\right)^2 r^2\cos^2\theta - 2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)r^2\sin\theta\cos\theta$$

$(\div r^2)$

$$\therefore \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta - 2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\cdot\sin\theta\cos\theta \quad (2)$$

$$(1) + (2) \Rightarrow \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

when $\sin^2\theta + \cos^2\theta = 1$