lecture (7)

Matrices

* **Powers of a matrix**

(props) If A is invertible matrix and n is a nonnegative integer, then

(1) $A^0 = I$, I is the identity matrix

(2) $A^n = A \cdot A \cdot A \ldots A$ (n factors)

(3) $A^{-n} = (A^n)^{-1} = (A^{-1})^n$ (n factors)

(4) $A^r A^s = A^{r+s}$

(5) $(A^r)^s = A^{rs}$

(6) $(A^{-1})^{-1} = A$

(7) $(KA)^{-1} = K^{-1}A^{-1} = \frac{1}{K}A^{-1}$, where K is a scalar

(8) $(A^T)^{-1} = (A^{-1})^T$, where A^T is the transpose of a matrix A

* **Transpose of a matrix**

If $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3} \Rightarrow A^T = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}_{3\times 2}$

(props)

(1) $(A^T)^T = A$ (2) $(AB)^T = B^T A^T$

(3) $(A+B)^T = A^T + B^T$ (4) $(KA)^T = KA^T$, K is a scalar

Ex Determine whether $A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$ is invertible if so, find its inverse and show that $(A^T)^{-1} = (A^{-1})^T$

Ans: $A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \Rightarrow \det(A) = 6(2) + 1(5) = 17$

$\det A \neq 0 \Rightarrow A$ is invertible and its inverse is

$A^{-1} = \frac{1}{17}\begin{bmatrix} 2 & 1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 2/17 & 1/17 \\ -5/17 & 6/17 \end{bmatrix}$

2

$$A^T = \begin{bmatrix} 6 & 5 \\ -1 & 2 \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{12+5}\begin{bmatrix} 2 & -5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 2/17 & -5/17 \\ 1/17 & 6/17 \end{bmatrix} \quad (1)$$

$$\therefore A^{-1} = \begin{bmatrix} 2/17 & 1/17 \\ -5/17 & 6/17 \end{bmatrix}$$

$$\therefore (A^{-1})^T = \begin{bmatrix} 2/17 & -5/17 \\ 1/17 & 6/17 \end{bmatrix} \quad (2)$$

$\therefore$ From (1), (2), we have $(A^T)^{-1} = (A^{-1})^T$ #

EX: A Matrix polynomial

Find $p(A)$ for $p(x) = x^2 - 2x - 3$ and $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

Ans:

$$p(A) = A^2 - 2A - 3I$$

$$= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Not that:

$$\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}_{2\times 2} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix}$$

3//

EX let A be a matrix $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ Compute

A^3, A^{-3}, $A^2 - 2A + I$

Ans:

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^{-3} = (A^3)^{-1} = \frac{1}{8-0}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1/8 & 0 \\ -7/2 & 1 \end{bmatrix}$$

$$A^2 - 2A + I$$

$$= \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

EX let A be an invertible matrix and suppose that inverse of $7A$ is $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$ find the matrix A

Ans:

$$(7A)^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$$

$$\therefore \frac{1}{7}A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -14 & 49 \\ 7 & -21 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = \frac{1}{14(21) - 7(49)}\begin{bmatrix} -21 & -49 \\ -7 & -14 \end{bmatrix}$$

$$\therefore A = \frac{-7}{-49}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3/7 & 1 \\ 1/7 & 2/7 \end{bmatrix}$$

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