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# Lecture (18)

* The linear Exponential Family p. (75) Textbook

**Def^n**

A random variable $X$ (discrete or continuous) has a distribution from the linear exponential family if its pdf may parameterized in terms of a parameter $\theta$ and expressed as

$$f(x;\theta)=\frac{p(x)\,e^{r(\theta)x}}{q(\theta)}$$

where $q(\theta)$ is the normalizing constant and $r(\theta)$ is the canonical parameter of the distribution. $p(x)$ is a fn only on $x$.

**Ex (1)** Show that the normal distribution is a member of the linear exponential family

**Ans:**

For $X\sim N(\theta,\sigma)$, Normal dist^n with mean $\theta$ and variance $\sigma$.

pdf $\Rightarrow f(x;\theta)=(2\pi\sigma)^{-1/2}\exp\left[\frac{-1}{2\sigma}(x-\theta)^2\right]$

$$=(2\pi\sigma)^{-1/2}\exp\left[\frac{-x^2}{2\sigma}+\frac{\theta}{\sigma}x-\frac{\theta^2}{2\sigma}\right]$$

$$\therefore f(x;\theta)=\frac{\left[(2\pi\sigma)^{-1/2}\exp\left(-\frac{x^2}{2\sigma}\right)\right]\exp\left(\frac{\theta}{\sigma}x\right)}{\exp\left(\frac{\theta^2}{2\sigma}\right)}$$

which of the form $f(x,\theta)=\frac{p(x)\,e^{r(\theta)x}}{q(\theta)}$

$\Rightarrow$

$p(x)=\left[(2\pi\sigma)^{-1/2}\exp\left(-\frac{x^2}{2\sigma}\right)\right]$, $r(\theta)=\frac{\theta}{\sigma}$ and

$q(\theta)=\exp\left(\frac{\theta^2}{2\sigma}\right)$ $\Rightarrow\therefore$ the normal dist^n is a member of the linear exp. family.

**Ex (2)** Show that the gamma dist^n is a member of the exponential family

For $X\sim \text{gamma}(\alpha,\theta)\Rightarrow f(x;\theta)=\frac{\theta^{-\alpha}x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)}$

clearly, $f(x;\theta)=\frac{p(x)\,e^{r(\theta)x}}{q(\theta)}=\frac{\left[x^{\alpha-1}/\Gamma(\alpha)\right]\cdot e^{-1/\theta\cdot x}}{\theta^{\alpha}}$

where $r(\theta)=-1/\theta$, $q(\theta)=\theta^{\alpha}$ and $p(x)=x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ the gamma dist^n is a member of the linear exponential family. #

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Ex(3) Find the mean and variance of the distribution defined by

$$f(x;\theta)=\frac{p(x)\,e^{r(\theta)x}}{q(\theta)}$$

Ans:

$$f(x;\theta)=\frac{p(x)\,e^{r(\theta)x}}{q(\theta)}$$

$$\Rightarrow \ln f(x;\theta)=\ln p(x)+r(\theta)x-\ln q(\theta)$$

$$\therefore \frac{1}{f}\frac{\partial f}{\partial\theta}=r'(\theta)x-\frac{1}{q(\theta)}\frac{\partial q}{\partial\theta}$$

$$\frac{\partial f(x;\theta)}{\partial\theta}=\left[r'(\theta)x-\frac{q'(\theta)}{q(\theta)}\right]f(x;\theta) \quad (1)$$

$$\Rightarrow \int\frac{\partial}{\partial\theta}f(x;\theta)\,dx=r'(\theta)\int x f(x;\theta)\,dx-\frac{q'(\theta)}{q(\theta)}\int f(x;\theta)\,dx$$

$$\therefore \frac{\partial}{\partial\theta}\left[\int f(x;\theta)\,dx\right]=r'(\theta)\int x f(x;\theta)\,dx-\frac{q'(\theta)}{q(\theta)}\int f(x;\theta)\,dx$$

$$\because \int f(x;\theta)\,dx=1,\quad \int x\,f(x;\theta)\,dx=E(X)$$

$$\therefore r'(\theta)E(X)=\frac{q'(\theta)}{q(\theta)} \Rightarrow E(X)=\mu(\theta)=\frac{q'(\theta)}{r'(\theta)\,q(\theta)} \quad (2)$$

(The mean)

To obtain the variance,

Substitute (2) in (1)

$$\frac{\partial f(x;\theta)}{\partial\theta}=\left[r'(\theta)x-r'(\theta)\mu(\theta)\right]f(x;\theta)$$

$$\frac{\partial f(x;\theta)}{\partial\theta}=r'(\theta)\left[x-\mu(\theta)\right]f(x;\theta) \quad (3)$$

$$\therefore \frac{\partial^2 f(x;\theta)}{\partial\theta^2}=r''(\theta)\left[x-\mu(\theta)\right]f(x;\theta)-r'(\theta)\mu'(\theta)f(x;\theta)+r'(\theta)\left[x-\mu(\theta)\right]\frac{\partial f(x;\theta)}{\partial\theta} \quad (4)$$

(3) in (4) $\Rightarrow$

$$\frac{\partial^2 f(x;\theta)}{\partial\theta^2}=r''(\theta)\left[x-\mu(\theta)\right]f(x;\theta)-r'(\theta)\mu'(\theta)f(x;\theta)+\left[r'(\theta)\right]^2\left[x-\mu(\theta)\right]^2 f(x;\theta)$$

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$\therefore \int \frac{\partial^2}{\partial \theta^2} f(x;\theta)\,dx = r''(\theta) \int [x-\mu(\theta)] f(x;\theta)\,dx$

$\quad - r'(\theta)\mu'(\theta) \int f(x;\theta)\,dx$

$\quad + [r'(\theta)]^2 \int [x-\mu(\theta)]^2 f(x;\theta)\,dx$ (5)

$S \because \int \frac{\partial^2}{\partial \theta^2} f(x;\theta)\,dx$

$= \frac{\partial^2}{\partial \theta^2} \underbrace{\int f(x;\theta)\,dx}_{1} = 0$

$\int [x-\mu(\theta)] f(x;\theta)\,dx$

$= \int x f(x;\theta)\,dx - \mu(\theta) \int f(x;\theta)\,dx$

$= E(X) - \mu(\theta) \cdot 1$

$= \mu(\theta) - \mu(\theta) = 0$

$\therefore (5) \Rightarrow 0 = 0 - r'(\theta)\mu'(\theta) \cdot 1 + [r'(\theta)]^2 \int [x-\mu(\theta)]^2 f(x;\theta)\,dx$

$\therefore \int [x-\mu(\theta)]^2 f(x;\theta)\,dx = \frac{\mu'(\theta)}{r'(\theta)} = Var(X)$

$\therefore$ The variance is $Var(X) = \sigma(\theta) = \frac{\mu'(\theta)}{r'(\theta)}$ (6)

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