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Lecture (31)

Bayesian & Bühlmann-Straub Models

EX 17.12 p. 414

Suppose that the number of claims N_j in year j for a group policyholder with (unknown) risk parameter θ and m_j individuals in the group is Poisson distributed with mean $m_j\theta$, that is, for $j=1,\dots,n$.

$$\Pr(N_j=x\mid\Theta=\theta)=\frac{(m_j\theta)^x e^{-m_j\theta}}{x!},\quad x=0,1,2,\dots$$

this result would be the case if per individual, the number of claims were independent Poisson distributed with mean θ. Determine the Bayesian expected number of claims for the m_{n+1} individuals to be insured in year $n+1$.

Ans:

$$\Pr(N_j=x\mid\Theta=\theta)=\frac{(m_j\theta)^x e^{-m_j\theta}}{x!},\quad x=0,1,2,\dots$$

The average number of claims per individual in year j is

$$X_j=\frac{N_j}{m_j},\quad j=1,\dots,n \quad (1)$$

$$\text{so, } f_{X_j\mid\Theta}(x_j\mid\theta)=\Pr\{N_j=m_jx_j\mid\Theta=\theta\}=\frac{(m_j\theta)^{m_jx_j}e^{-m_j\theta}}{m_jx_j!},\quad j=1,\dots,n \quad (2)$$

where $\Theta\sim$ gamma (α,β)

$$\pi(\theta)=\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha},\quad \theta>0 \quad (3)$$

for $X\sim$ gamma (α,β) $\Rightarrow$ $f(x)=\frac{(x/\beta)^\alpha e^{-x/\beta}}{x\Gamma(\alpha)},\ x>0$, $x\to\theta$

The posterior distribution function is

$$\pi_{\Theta\mid X}(\theta\mid x)\propto\left[\prod_{j=1}^{n} f_{X_j\mid\Theta}(x_j\mid\theta)\right]\pi(\theta)$$

$$\propto\left[\prod_{j=1}^{n}\theta^{m_jx_j}e^{-m_j\theta}\right]\theta^{\alpha-1}e^{-\theta/\beta}$$

$$=\theta^{\alpha+\sum_{j=1}^{n}m_jx_j-1}e^{-\theta\left(\beta^{-1}+\sum_{j=1}^{n}m_j\right)} \quad (4)$$

which is proportional to gamma density with parameters

$$\alpha_*=\alpha+\sum_{j=1}^{n}m_jx_j \text{ and } \beta_*=\left(1/\beta+\sum_{j=1}^{n}m_j\right)^{-1}$$

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Now, $E(X_j|\Theta=\theta) = E\left(\frac{N_j}{m_j}\middle|\Theta=\theta\right)$, See (1)

$$= \frac{1}{m_j} E(N_j|\Theta=\theta)$$

$\because N_j|\Theta \sim \text{poisson}(m_j\theta)$ given, See (2)

$$\therefore E(X_j|\Theta=\theta) = \frac{1}{m_j}(m_j\theta) = \theta$$

Thus $\mu_{n+1}(\theta) = E(X_{n+1}|\Theta=\theta) = \theta$ hypothetical mean,

and $\mu_{n+1} = E(X_{n+1}) = E[\mu_{n+1}(\Theta)]$ expected value of the hypoth. mean.

$= E[\Theta] = \alpha\beta$, where $\Theta \sim \text{gamma}(\alpha,\beta)$

Not that $E[E(X_{n+1}|\Theta=\theta)] = E(X_{n+1})$

$$\because E(X_{n+1}|X=x) = \int_0^\infty \mu_{n+1}(\theta)\,\pi_{\Theta|X}(\theta|x)\,d\theta$$

$$= E\left[\mu_{n+1}(\Theta)\,\middle|\,X=x\right]$$

$$\therefore E(X_{n+1}|X=x) = E(\Theta|X=x) = \alpha_*\beta_*,\ \text{See (4)}$$

So, we can define the Bayesian premium by using partial Credibility as

$$E(X_{n+1}|X=x) = Z\bar{x} + (1-Z)\mu_{n+1}$$

where $Z = \frac{m}{m+\beta^{-1}}$, $m = \sum_{j=1}^{n} m_j$ is the total number of lives that observed

$\bar{x} = m^{-1}\sum_{j=1}^{n} m_j x_j$ and $\mu_{n+1} = \alpha\beta$.

$\therefore$ The total Baysian number of claims for m_{n+1} individuals in the group for the next year would be

$$m_{n+1}E(X_{n+1}|X=x). \quad \#$$

Not that If $m_j=1$, then $X_j = N_j$ for $j=1,2,\cdots,n$.
and Consequently

$$E(X_{n+1}|X=x) = Z\bar{x} + (1-Z)\mu$$

where

$$Z = \frac{n}{n+\beta^{-1}},\quad \bar{x} = n^{-1}\sum_{j=1}^{n} x_j \text{ and } \mu = \alpha\beta.$$

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* The Bühlmann – Straub Model

The Bühlmann model is the simplest of credibility models because it depends on the past claims experience of a policyholder, but it doesnot allow for variations in exposure or size.
So, we consider the following generalization of the Bühlmann model. Assume that $X_1, \ldots, X_n$ are independent conditional on Θ with common mean (as before)

$$\mu(\theta) = E(X_j \mid \Theta = \theta)$$

but with conditional variances

$$Var(X_j \mid \Theta = \theta) = \frac{v(\theta)}{m_j},$$

where m_j is a known constant measuring exposure, and it's proportional to the size of the risk. m_j could be the number of months the policy was inforce in past year j, or the number of individuals in the group in past year j, or the amount of premium income for the policy in past year j.
As in the Bühlmann Model, let

$$\mu = E[\mu(\Theta)], \quad v = E[v(\Theta)],$$

$$a = Var[\mu(\Theta)] \text{ and } k = \frac{v}{a}$$

Let $m = m_1 + m_2 + \cdots + m_n$ be the total exposure, the credibility premium for Bühlmann – Straub Model is given as

$$Z\bar{X} + (1-Z)\mu$$

$$\text{where } Z = \frac{m}{m+k} \quad \text{and } \bar{X} = \sum_{j=1}^{n} \frac{m_j}{m} X_j$$

EX 17.17 p.424

As in EX 17.12, assume that in year j there are N_j claims from m_j policies, $j = 1, 2, \ldots, n$. An individual policy has a poisson distribution with parameter Θ, and the parameter itself has a gamma dist^n with parameters α and β. Determine the Bühlmann – Straub estimate of the number of claims in year $n+1$ if there will be m_{n+1} policies.

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Ans:

Let $X_j = N_j/m_j$, see EX 17.12

$\because N_j|\Theta$ has a Poisson dist$\underline{n}$ with mean $m_j\Theta$,

$$E(X_j|\Theta) = E\left(\frac{N_j}{m_j}\Big|\Theta\right) = \frac{1}{\cancel{m_j}}\cancel{m_j}\Theta = \Theta = \mu(\Theta)$$

and $Var(X_j|\Theta) = Var\left(\frac{N_j}{m_j}\Big|\Theta=\theta\right)$

$$= \frac{1}{m_j^2} Var(N_j|\Theta) = \frac{\cancel{m_j}\Theta}{m_j^{\cancel{2}}}$$

$$= \frac{\Theta}{m_j} = v(\Theta)/m_j$$

$\Rightarrow \mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypoth. means, where $\Theta \sim$ gamma (α,β),

$v = E[v(\Theta)]$ the expected value of process variance $= E(\Theta) = \alpha\beta$,

and $a = Var(\Theta) = \alpha\beta^2$ the variance of hypoth. means.

$$\therefore K = \frac{v}{a} = \frac{\cancel{\alpha\beta}}{\cancel{\alpha\beta^2}} = \frac{1}{\beta},$$

$$Z = \frac{m}{m+k} = \frac{m}{m+1/\beta} = \frac{m\beta}{m\beta+1}$$

So, the Bühlmann–Straub estimate for one policyholder is

$$P_c = \frac{m\beta}{m\beta+1}\bar{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu$$

$$= \frac{m\beta}{m\beta+1}\bar{X} + \frac{1}{m\beta+1}\alpha\beta$$

where $\bar{X} = m^{-1}\sum_{j=1}^{n} m_j X_j$

For year $n+1$, the estimate is $m_{n+1}P_c$, matching the answer to EX 17.12.