PHYS-505/551

Algebraic Theory of Angular Momentum

Lecture-2

Introduction-a

- Classicaly, the angular momentum of a particle is defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad (2.1)$$

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x \qquad (2.2)$$

- The corresponding quantum operators are obtained by the standard prespriction

$$p_x \to -i\hbar\partial / \partial x, \; p_y \to -i\hbar\partial / \partial y, \; p_z \to -i\hbar\partial / \partial z \quad (2.3)$$

Introduction-b

- The components of the angular momentum operator satisfy the following relation:

$$\left[l_z,\, l_x\right] = i\hbar l_y,\quad \left[l_x,\, l_y\right] = i\hbar l_z,\quad \left[l_y,\, l_z\right] = i\hbar l_x \qquad (2.4)$$

- From which we can show that we get

$$\left[\mathbf{l}^2,\, l_i\right] = 0, \qquad i = 1,2,3 \equiv x, y, z \qquad (2.5)$$

- This physically means that from the four physical quantities $l_x,\, l_y,\, l_z,\, \mathbf{l}^2$ we can measure simultaneously only $\mathbf{l}^2$ and one of the components $l_x,\, l_y,\, l_z$

For proving the above relations you may need the following commutator relations: [A, BC]=B[A,C]+[A, B]C

$$\left[x, p_x\right] = i\hbar,\ \left[y, p_y\right] = i\hbar,\ \left[z, p_z\right] = i\hbar$$

The algebraic theory of angular momentum-a

- The problem here is quite similar (but not the same) to the case of a harmonic oscillator. We seek to solve the eigenvalue equation of the operator l_z:

$$l_z \left| l, m \right\rangle = m\hbar \left| l, m \right\rangle \qquad (2.6)$$

- That means to find all the allowed values of the projection of the angular momentum vector along the z axis - but provided that simultaneously this vector has a fixed magnitude. This means that simultaneously the following relation will be satisfied:

$$\mathbf{l}^2 \left| l, m \right\rangle = l\left(l+1\right)\hbar^2 \left| l, m \right\rangle \qquad (2.7)$$

The algebraic theory of angular momentum-b

- The algebraic theory of angular momentum is based on two new operators l_+, l_- which - at analogy to the simple harmonic oscillator creation and destruction operators $a^\dagger, a$ - "raise" and "lower" the eigenvalues of the angular momentum component l_z

  but do not change the eigenvalues of $\mathbf{l}^2$ i.e.

$$\left[l_\pm, \mathbf{l}^2\right] = 0 \qquad (2.8)$$

The algebraic theory of angular momentum-c

- These operators are given by the following relations:

$$l_{\pm} = l_x \pm i l_y \qquad (2.9)$$

- And they satisfy the following relations:

$$\left[l_z,\, l_+\right] = \hbar l_+, \quad \left[l_z,\, l_-\right] = -\hbar l_- \qquad (2.10)$$

The algebraic theory of angular momentum-d

- The action of the raising and lowering operators on a given state $|lm\rangle$ is given by the following relations:

$$l_+|lm\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\,|l,m+1\rangle \qquad (2.11a)$$

$$l_-|lm\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,|l,m-1\rangle \qquad (2.11b)$$

The algebraic theory of angular momentum-e

- An interesting point of the above relations is the problem of "termination" of the production of new eigenstates with the help of the raising and lowering operators.
- Termination is physically forced because for a given magnitude of the angular momentum, the projection (positive or negative) along an axis cannot be larger than the magnitude itself.
- By examining the above relations we can conclude that $m_{\max} = l$ and $m_{\min} = -l$

The algebraic theory of angular momentum-g

- The major advantage of the algebraic theory of angular momentum is that allows the calculations of various physical quantities using just algebraic techniques and avoiding the use of the complicated eigenfunctions, integrations etc.
- The expressions of the operators l_x, l_y with the help of raising and lowering operators given below, may be very helpful in solving problems.

$$l_x = \frac{l_+ + l_-}{2}, \quad l_y = \frac{l_+ - l_-}{2i} \qquad (2.12)$$

The algebraic theory of angular momentum-h

- We must point out that the eigenvalues of l_z go from $m_{\max} = l$ to $m_{\min} = -l$ in N integer steps. This means that

$$l = -l + N \Rightarrow 2l = N \Rightarrow l = N/2$$

- So l must be an *integer* or *half-integer*
- It is also obvious that there are $2l+1$ different values of m.

The algebraic theory of angular momentum: Matrix representation-a

- An operator can be represented in matrix form; this representation depends on the basis vectors (eigenvector) that we choose. For an angular momentum operator we usually use the standard basis $|lm\rangle$, so every matrix element A_{ij} that represent the operator satisfies $A_{ij} = \langle li|A|lj\rangle$. Thus for every *l=const*, we can write a $(2l+1)\times(2l+1)$ matrix for the operators $\mathbf{l}^2, l_x, l_y, l_z$.

The algebraic theory of angular momentum: Matrix representation-b

- We can show the following relations:

$$\left(\mathbf{l}^2\right)_{ij} = \left\langle li\left|\mathbf{l}^2\right|lj\right\rangle = l\left(l+1\right)\hbar^2\delta_{ij}, \qquad \left(l_z\right)_{ij} = \left\langle li\left|l_z\right|lj\right\rangle = j\hbar\delta_{ij}$$

$$\left(l_x\right)_{ij} = \left\langle li\left|l_x\right|lj\right\rangle = \frac{\hbar}{2}\left[\sqrt{\left(l-m\right)\left(l-m+1\right)}\delta_{i,j+1} + \sqrt{\left(l+m\right)\left(l-m+1\right)}\delta_{i,j-1}\right]$$

$$\left(l_y\right)_{ij} = \left\langle li\left|l_y\right|lj\right\rangle = \frac{\hbar}{2}\left[\sqrt{\left(l-m\right)\left(l+m+1\right)}\delta_{i,j+1} - \sqrt{\left(l+m\right)\left(l-m+1\right)}\delta_{i,j-1}\right]$$

(2.13)

Angular momentum and matrix representation-a

- Let $|\psi_O>$ be a state vector of a system in a certain coordinate system O. To represent the state vector in another coordinate system S we define the rotation operator U_R such that the state vector $|\psi_S>$ in S is given

$$|\psi_S\rangle = U_R|\psi_O\rangle \qquad (2.14)$$

Angular momentum and matrix representation-b

- If the system S is obtained by the rotation, with an angle θ, of the sytem O around an axis in the direction of the unitary vector **n** then

$$U_R(\theta,\mathbf{n}) = \exp\left(-\frac{i}{\hbar}\theta\mathbf{n}\cdot\mathbf{l}\right) \qquad (2.15)$$

- This is the reason for which we call the angular momentum **l** as the *generator of rotation*. For any observable of the system O represented by the operator A_O we can get its operator in system S by the relation

$$A_S = U_R A_O U_R^{\dagger}, \qquad A_O = U_R^{\dagger} A_S U_R \qquad (2.16)$$

Some final important comments-a

- The algebraic theory which we developed in this lecture was based solely on commutation relations of the components of **l** and **nothing else**.
- This means that the theory of this lecture could be used **unchanged** for any other kind of "angular momentum" (for example the spin - next lecture).

Some final important comments-b

- The algebraic theory of a angular momentum has an advantage over the method of separating variables which you used in your undergraduate studies to solve the Schroedinger Equation and to study the hydrogen atom: It permits l (and hence also m) to take on also *half-integer* values, whereas the separation of variables yielded eigenfunctions only for *integer values*. These half-integer values are of profound importance. They will lead us to a new form of angular momentum: the spin.