PHYS-505/551

The Spin

Lecture-3

Introduction-a

- We have seen in Lecture-2 that the algebraic theory of the angular momentum has revealed the possibility of existence of half integer values for the quantum number of the angular momentum magnitude.
- Today we know that this *inherent angular momentum* - the well know **spin**- trully exists and constitutes a fundamental property of the particles, which is more important than their mass or charge. Spin is related to the famous *Pauli Exclusion Principle.*
- Particles with half-integer spin behave totally different than particles with integer spin.

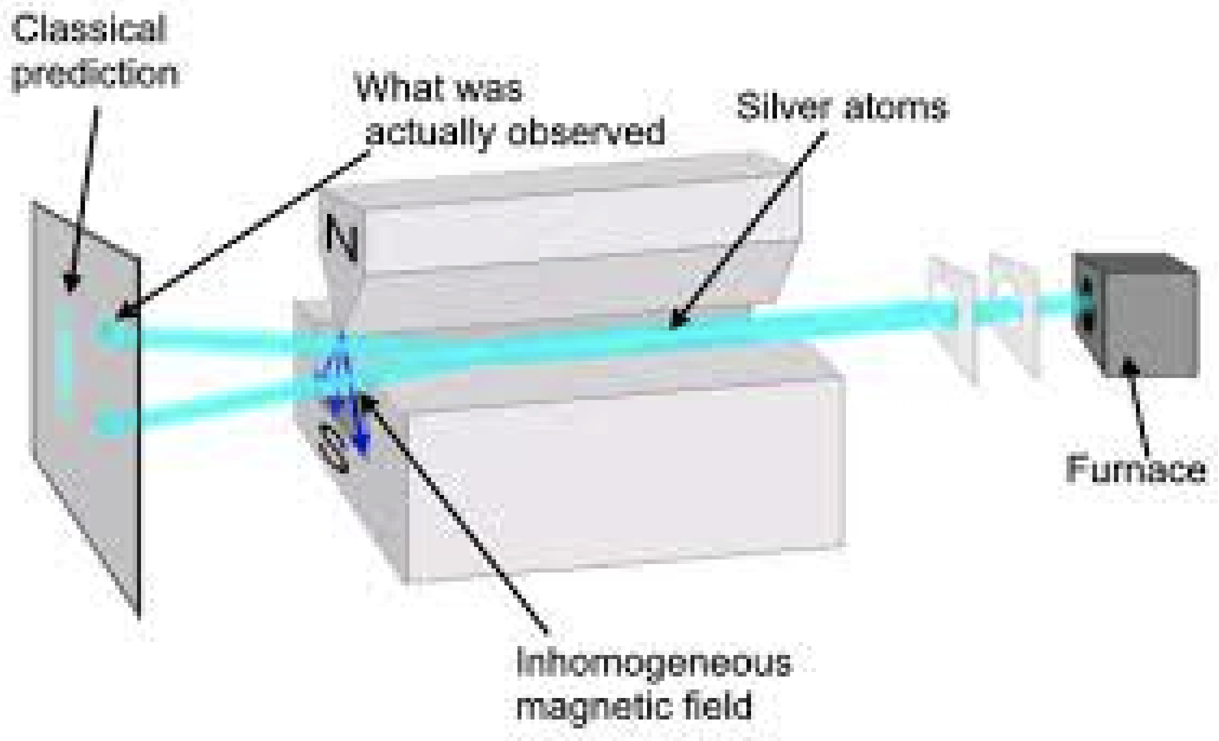
Experimental and physical evidence for the existence of electron's spin-a

- The *Periodic Table* is the classical example for the existence of electron's spin. The existence of atomic orbitals with a fixed capacity in electrons is an exhibition of Pauli's principle. The fact that in the orbital with n=1 we can accommodate only two electrons could be explained only with the spin assumption.
- But the periodic table in combination with Pauli principle provides only an indirect evidence for the spin of electron.

Experimental and physical evidence for the existence of electron's spin-b

- The confirmation for the existence of spin came from the famous *Stern-Gerlach Experiment*. The basic idea of this experiment is the following:
- We send a beam of hydrogen atoms (in their ground state) to cross a region of a non-uniform static magnetic field. The force exerted by the field on the atoms is different for different spin orientations, thus will split the beam in as many components as the number of the different spin orientations. In the case of hydrogen the beam will split in two smaller beams travelling in opposite directions.

Experimental and physical evidence for the existence of electron's spin-c



The algebraic theory of spin-a

- From all the experimental evidence about spin the essential for us is: *spin is also an angular momentum*. It has the same experimental *signature* as *orbital angular momentum* **l**. The only difference is the existence of half-integer values.
- Moreover the magnitude of spin is fixed to a certain value which is a permanent characteristic of the particle!

The algebraic theory spin-b

- The algebraic theory of spin is also based on the commutative properties of its components, i.e.

$$\left[s_z,\ s_x\right] = i\hbar s_y,\quad \left[s_x,\ s_y\right] = i\hbar s_z,\quad \left[s_y,\ s_z\right] = i\hbar s_x \qquad (3.1)$$

in combination with the condition

$$\mathbf{s}^2 = s\left(s+1\right)\hbar^2 \qquad (3.2)$$

where s is the quantum spin number - or the spin magnitue.

The algebraic theory spin-c

- According to the algebraic theory of previous lecture the action of the operators s_x, s_y, s_z, $\mathbf{s}^2$ on the spin eigenstates $\left|s, m_s\right\rangle$ is given by

$$s_z \left|s, m_s\right\rangle = m_s \hbar \left|s, m_s\right\rangle \qquad (3.3)$$

$$\mathbf{s}^2 \left|s, m_s\right\rangle = s\left(s+1\right)\hbar^2 \left|s, m_s\right\rangle \qquad (3.4)$$

The algebraic theory spin-d

- The action of the raising and lowering operators on a given state $|s, m_s\rangle$ is given by the following relations:

$$s_+ |s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|s, m_s+1\rangle \qquad (3.5)$$

$$s_- |s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s, m_s-1\rangle \qquad (3.6)$$

$$s_\pm = s_x \pm i s_y \qquad (3.7)$$

A system with spin s=1/2-a

- In a system with spin $s = 1/2$ there are two basic states of spin orientation (the **spin-up** and the **spin-down**) which are described by the state vectors: $|1/2, 1/2\rangle \equiv |+\rangle$, $|1/2, -1/2\rangle \equiv |-\rangle$
- These vectors (or spinors) are simultaneously the basis vectors of a two-dimensional space which includes all the vectors of the form:

$$|\psi\rangle = a|+\rangle + b|-\rangle \qquad (3.8)$$

s=1/2 is the spin of the partcles that make up ordinary matter (protons, neutrons and electrons) as well as all quarks and all leptons.

A system with spin s=1/2-b

- Where a and b are in general complex numbers which are related to the corresponding probabilities of finding the particle in the up and down states:

$$P_+ = |a|^2, \quad P_- = |b|^2 \qquad (3.9)$$

$$|a|^2 + |b|^2 = 1 \qquad (3.10)$$

$$a = \langle + | \psi \rangle, \quad b = \langle - | \psi \rangle, \quad (3.11)$$

$$\langle + | - \rangle = \langle - | + \rangle = 0, \quad \langle + | + \rangle = \langle - | - \rangle = 1. \qquad (3.12)$$

The matrix representation of spin-a

- As we have said we can, instead of state vectors, to use matrices to represent spin states. In this representation the spin-up and spin down states are represented by the following column vectors:

$$X_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (3.13)$$

The matrix representation of spin-b

- The next step is the representation of the operators s_x, s_y, s_z in matrix form. These matrices are as follows:

$$s_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad s_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad s_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.14}$$

$$s_+ = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_- = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.15}$$

The Pauli matrices-a

- The Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ are matrices which are related to the spin matrices as follows:

$$s_i = \frac{\hbar}{2}\sigma_i, \ \left(i \equiv x, y, z\right) \qquad (3.16)$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (3.17)$$

Pauli matrices are all Hermitian.

The Pauli matrices-b

- Spin Matrices

$$s_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$s_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$s_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices-c

$$\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$$

$$\mathrm{Tr}\sigma_x = \mathrm{Tr}\sigma_y = \mathrm{Tr}\sigma_z = 0$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

$$\sigma_x\sigma_y = -\sigma_y\sigma_x + \text{ cyclic permutations}$$

$$\sigma_x\sigma_y = i\sigma_z + \text{ cyclic permutations} \tag{3.18}$$

A wrong interpretation of spin

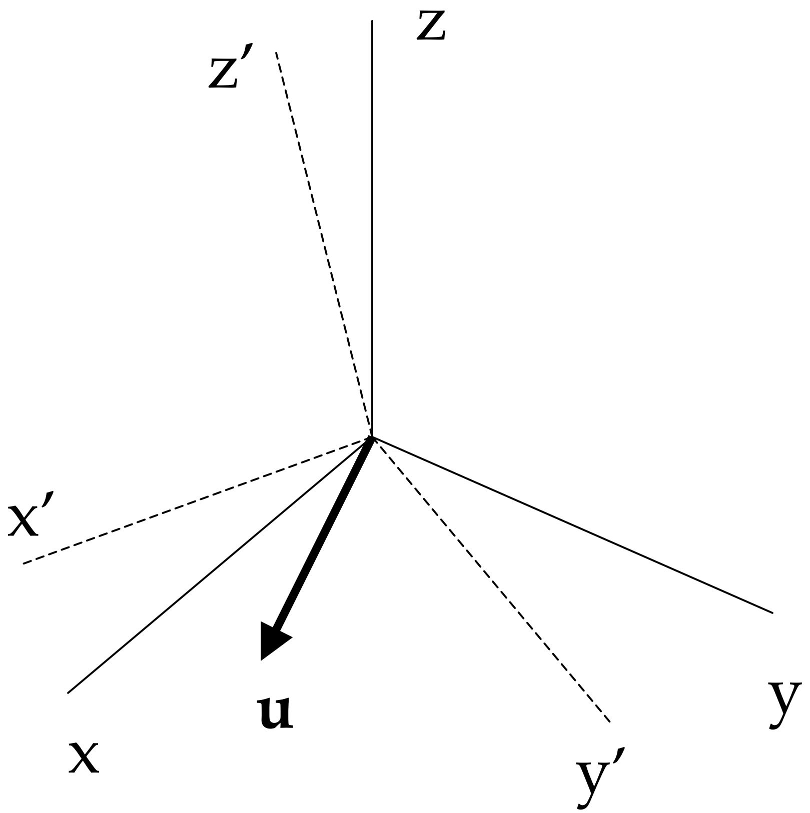
- So far all the physical quantities we have encountered in QM have their classical counterparts. What is the classical counterpart of spin?
- Usually spin is related to an angular momentum of a particle around an axis that passes through it. (Something similar to the rotation of the Earth around its axis)
- This picture is absolutely wrong! A) If the spin was a rotation around the axis then the speed of the particle in its equator would be far larger than the speed of light. B) In such a case during collision between the particles we could see the excitation of states with higher spin than $s = 1/2$. This never happens! Spin is a constant identity property for a particle like its mass and charge.
- Spin has nothing to do with motion in space. That' why in its description we do not encounter the variables: r, θ, ϕ!

Rotations in spin space-a

- To find the representation of a state $|a'\rangle$ in a given coordinate system that is rotated at an angle θ around an axis in the direction of the unit vector **u** we compute:

$$|a'\rangle = \exp\left(-\frac{i}{\hbar}\theta \mathbf{u}\cdot\mathbf{S}\right)|a\rangle$$

(3.19)



Rotations in spin space-b

- The rotation matrix is given by:

$$U_R = \exp\left(-\frac{i}{\hbar}\theta \mathbf{u}\cdot\mathbf{S}\right) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2)e^{-i\phi} \\ \sin(\theta/2)e^{i\phi} & \cos(\theta/2) \end{pmatrix} \quad (3.20)$$

- Note that for ϕ=0 (rotation around z axis) we get:

$$U_R = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \quad (3.21)$$

which corresponds to a rotation $\theta/2$ around z axis.

Mathematical Supplement-a

- Let an operator A which represents a quantum mechanical physical quantity. For this operator corresponds a matrix which represents the operator in a given ortho-unitary basis $|n>$, with elements A_{nm} which are given by:

$$A_{nm} = \langle n | A | m \rangle$$

Mathematical Supplement-b

- **How we construct the matrix which correspond to an operator?**
- *We put as columns the vectors* $A|1\rangle,\ A|2\rangle,\ldots A|m\rangle$ *which are taken by the action of the operator on the basis vectors*

$$A = \begin{pmatrix} . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ A|1\rangle & A|2\rangle & . & . & . & A|m\rangle \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \end{pmatrix}$$