PHYS-505/551

The addition of angular momenta

Lecture-4

Introduction-a

- When we seek to find the total angular momentum in the hydrogen atom we must add the orbital angular momentum **l** with the spin s. For the same purpose in multielectron atoms we need to add the orbital angular momenta and spin of all the electrons.
- The total angular momentum **j** is defined the vector sum orbital angular momentum **l** and the spin **s** as follows:

$$\mathbf{j} = \mathbf{l} + \mathbf{s} \qquad (4.1)$$

Introduction-b

- But now the vectors **l** and **s** are quantum vectors which obey the following permutation relations:

$$[l_z, l_x] = i\hbar l_y, \quad [l_x, l_y] = i\hbar l_z, \quad [l_y, l_z] = i\hbar l_x \qquad (4.2)$$

$$[s_z, s_x] = i\hbar s_y, \quad [s_x, s_y] = i\hbar s_z, \quad [s_y, s_z] = i\hbar s_x \qquad (4.3)$$

- The first step is to show that the vector **j** is also an angular momentum, that means it satisfies similar relations as (4.2) and (4.3).

The total angular momentum-a

- Since the quantity **j**=**l**+**s** is indeed an angular momentum it must obey the following two foundamental relations:

$$\mathbf{j}^2\left|j,m_j\right\rangle = j\left(j+1\right)\hbar^2\left|j,m_j\right\rangle \qquad (4.4)$$

$$j_z\left|j,m_j\right\rangle = m_j\hbar\left|j,m_j\right\rangle \qquad (4.5)$$

- Where m_j for a given j will get $2j+1$ values

$$m_j = -j,\ \ldots\ ,+j \qquad (4.5)$$

The total angular momentum-b

- Now the question comes naturally:
  What are the possible values of j for given l and s?
- The answer is: *For given l and s the angular momentum j takes the values:*

$$j = |l - s|,\ \underbrace{\ldots\ldots}_{\text{unit step}}\ ,\ l + s$$

The proof is given in the class

The eigenstates of the total angular momentum-a

- With the following examples we show the idea of constructing the total angular momentum eigenstates:
- A) Construct the states of definite total angular momentum of a hydrogen atom at the state 2*p*.
- B) Construct the state of definite total spin for two particles with spin 1/2 each.

Examples will be solved in class

The eigenstates of the total angular momentum-b

- In general let's consider two different angular momenta $\mathbf{j}_1$, $\mathbf{j}_2$. These momenta can be angular momenta relating two different particles or angular momenta relating to one particle (for example, orbital angular momentum and spin).
- These two momenta act in different state spaces, so that all their components are commuting with one another. The individual states of $\mathbf{j}_1$, $\mathbf{j}_2$ will be denoted, as usual,

$$\left| j_1 m_1 \right\rangle, \ \left| j_2 m_2 \right\rangle$$

The eigenstates of the total angular momentum-c

- For this states we have the usual properties:

$$\begin{cases} \mathbf{j}_1^2 \left| j_1 m_1 \right\rangle = \hbar^2 j_1 \left(j_1 + 1 \right) \left| j_1 m_1 \right\rangle \\ j_{1z} \left| j_1 m_1 \right\rangle = \hbar m \left| j_1 m_1 \right\rangle \end{cases}$$

(similarly for the particle 2)

- The state space of the compound system is obtained by taking the direct product (or tensor product) of the individual state space of the two angular momenta:

$$\left| j_1 m_1 \right\rangle \otimes \left| j_2 m_2 \right\rangle = \left| j_1 j_2 \ ; \ m_1 m_2 \right\rangle \equiv \left| m_1 m_2 \right\rangle$$

The eigenstates of the total angular momentum-d

- For fixed j_1, j_2, m_1 and m_2 have the values (integer or half-integers):

$$\begin{cases} m_1 = -j_1, -j_1 + 1, \ \dots j_1 \\ m_2 = -j_2, -j_2 + 1, \ \dots j_2 \end{cases}$$

- The state space of the compound system is a $(2j_1+1)(2j_2+1)$ -dimensional space.
- The states $\left| m_1 m_2 \right\rangle$ are, according to their construction, eigenstates of the operators

$$\left\{ \mathbf{j}_1^2, \ \mathbf{j}_2^2, \ j_{1z}, \ j_{2z} \right\}$$

The eigenstates of the total angular momentum-e

- In the absence of interaction between $\mathbf{j}_1$, $\mathbf{j}_2$, the operators $\mathbf{j}_1$, $\mathbf{j}_2$ commute with the total Hamiltonian and thus $|j_1 m_1\rangle, |j_2 m_2\rangle$ are also eigenstates of the system. But what happens if the there is an interaction between $\mathbf{j}_1$, $\mathbf{j}_2$?
- In this case $\mathbf{j}_1$, $\mathbf{j}_2$ are not conserved but $\mathbf{j} = \mathbf{j}_1+\mathbf{j}_2$ is conserved. It is better then to transform to an eigenstate basis of the operators

$$\left\{\mathbf{j}_1^2,\ \mathbf{j}_2^2,\ \mathbf{J}^2,\ J_z\right\}$$

The eigenstates of the total angular momentum-f

- The eigenstates in this basis will be denoted by $\left| j_1\, j_2\, J\, M \right\rangle \equiv \left| J\, M \right\rangle$ and satisfy

$$\begin{cases} \mathbf{J}^2 \left| JM \right\rangle = \hbar^2 J\left(J+1\right)\left| JM \right\rangle \\ J_z \left| JM \right\rangle = \hbar M \left| JM \right\rangle \end{cases}$$

$$J = \left| j_1 - j_2 \right|,\ \left| j_1 - j_2 \right| + 1, \underbrace{\ldots\ldots}_{\text{unit step}}, j_1 + j_2$$

$$M = -J,\ -J+1,\ \ldots,\ J$$

The Clebsch-Gordan coefficients-c

- The two sets of orthonormal states $|m_1 m_2\rangle$ and $|JM\rangle$ are related by a unitary transform; that is we can write $|JM\rangle$ in terms of $|m_1 m_2\rangle$ as follows

$$|JM\rangle = \sum_{m_1, m_2} \langle m_1 m_2 \mid JM\rangle |m_1 m_2\rangle$$

- The terms $c_{m_1 m_2} = \langle m_1 m_2 \mid JM\rangle$ are known as *Clebsch-Gordan coefficients* .
- This means that, from the linear combination to get a state with not only a definite *M* but also with a definite *J*

The Clebsch-Gordan coefficients-d

- It is possible to obtain a general expression for the C-G coefficients. However it is simpler to construct the coefficients for particular cases. They can be calculated by successive applications of $J_{\pm} = J_x \pm iJ_y$ on the vectors $\left|JM\right\rangle$ as follows:

$$\begin{cases} J_{\pm}\left|JM\right\rangle = \hbar\sqrt{J(J+1) - M(M+1)}\left|J, M \pm 1\right\rangle \\ J_{1\pm}\left|m_1 m_2\right\rangle = \hbar\sqrt{J_1(J_1+1) - m_1(m_1+1)}\left|m_1 \pm 1, m_2\right\rangle \end{cases}$$

- Together with the relation:

$$\left|J = J_1 + J_2, M = \pm\left(j_1 + j_2\right)\right\rangle = \left|m_1 = \pm j_1,\ m_2 = \pm j_1\right\rangle$$

Properties of the Clebsch-Gordan coefficients-a

$$\langle m_1, m_2 | JM \rangle = 0 \qquad \text{unless} \quad M = m_1 + m_2$$

$$\langle m_1, m_2 | JM \rangle = \text{ is real}$$

$$\sum_{m_1=-j_1}^{m_1=j_1} \sum_{m_2=-j_2}^{m_2=j_2} \langle JM | m_1, m_2 \rangle \langle m_1, m_2 | J' M' \rangle = \delta_{JJ'} \delta_{MM'}$$

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-j}^{J} \langle m_1, m_2 | JM \rangle \langle JM | m_1', m_2' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

Properties of the Clebsch-Gordan coefficients-b

$$\sqrt{J(J+1)-M(M+1)}\left\langle m_1 m_2 \middle| J, M+1 \right\rangle =$$
$$\sqrt{j_1(j_1+1)-m_1(m_1+1)}\left\langle m_1 \mp 1, m_2 \middle| JM \right\rangle + \sqrt{j_2(j_2+1)-m_2(m_2+1)}\left\langle m_1, m_2 - 1 \middle| JM \right\rangle$$

$$\left\langle m_2 m_1 \middle| JM \right\rangle = \left(-1\right)^{j_1+j_2-J}\left\langle m_1 m_2 \middle| JM \right\rangle$$
$$\left\langle -m_1, -m_2 \middle| J, -M \right\rangle = \left(-1\right)^{j_1+j_2-J}\left\langle m_1 m_2 \middle| JM \right\rangle$$