CEN352
Digital Signal Processing

By

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Lecture No. 3

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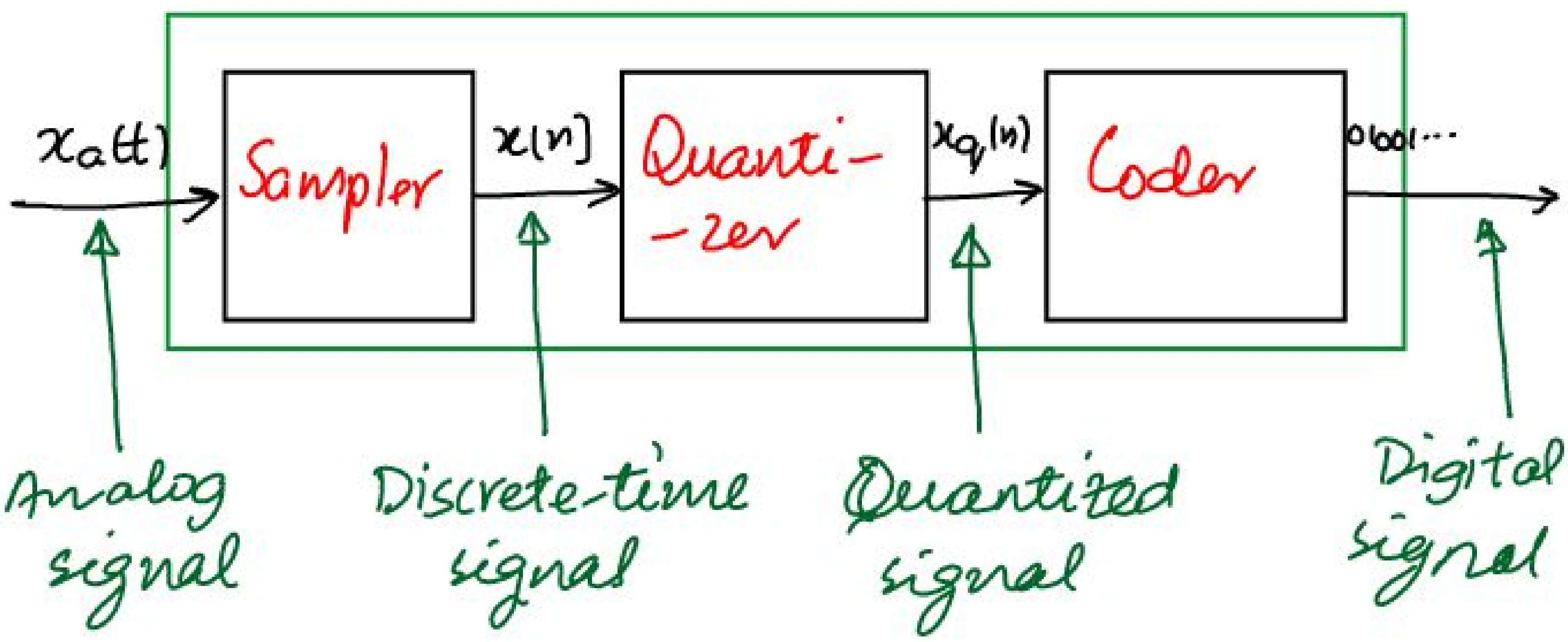
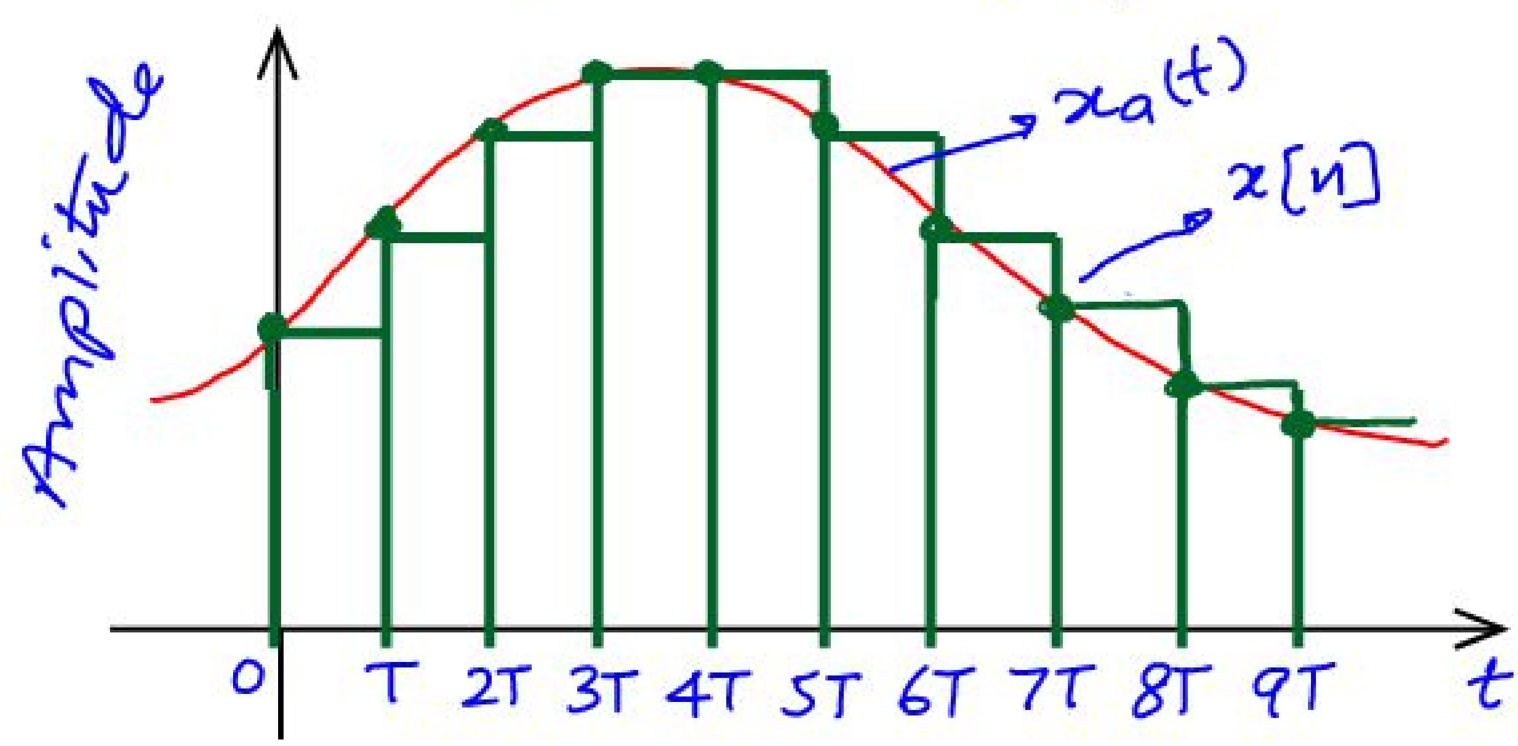
September, 2012 - January, 2013

Basic components of an analog-to-digital Converter



Sampling: The process of converting a continuous -time signal into a discrete-time signal by taking "samples" of the continuous-time signal at discrete-instants of time.



If $x_a(t)$ is the analog input to the sampler, the output is

$$x[n] \equiv x_a(nT)$$

where T is called the sampling time (it is the time span between two consecutive samples.

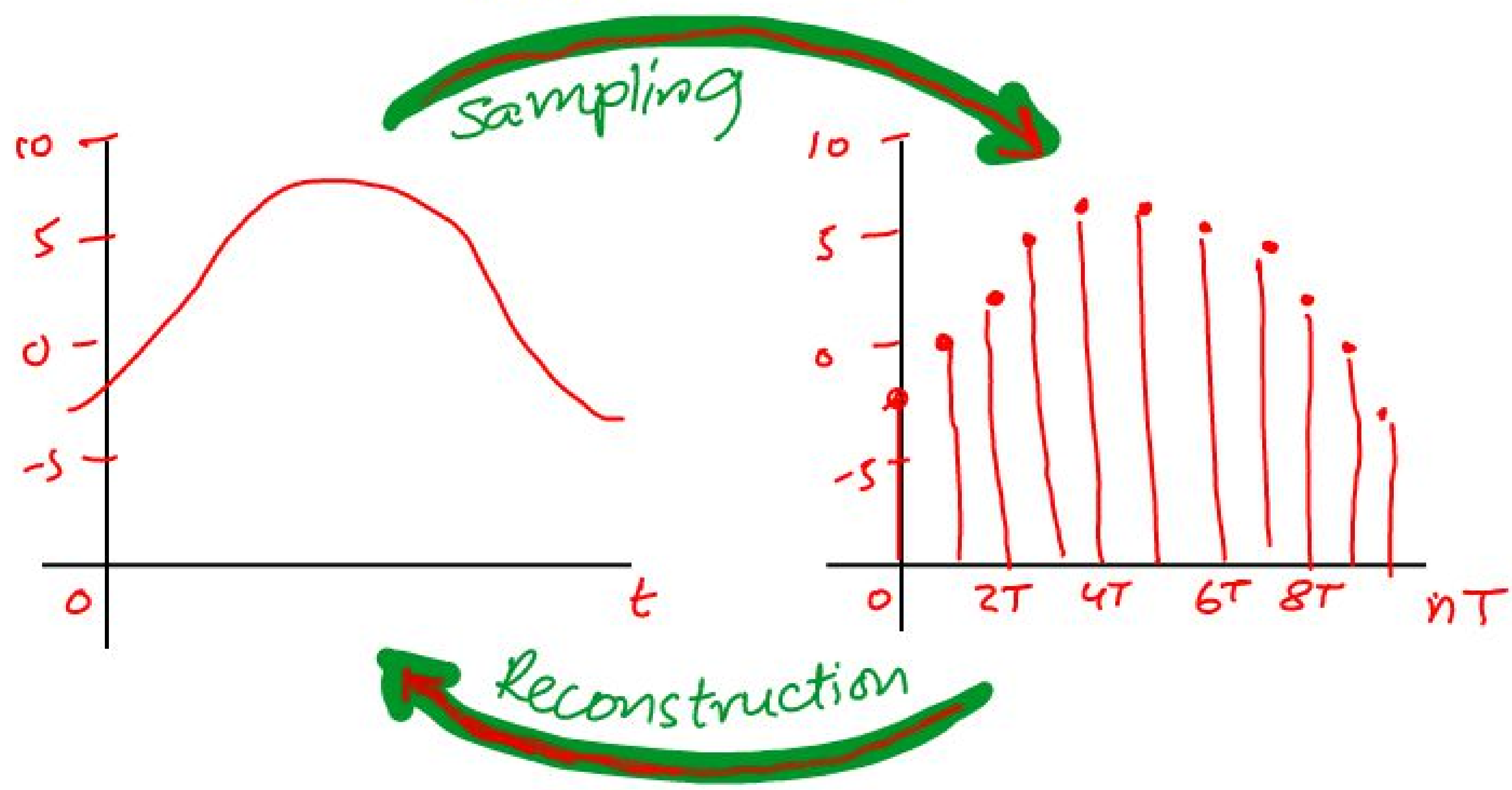
The sampling rate F_s is the number of samples taken in unit time i.e.

$$F_s = \frac{1}{T} \text{ samples per second (Hz)}$$

For example; if the sampling time is

$$T = 125 \text{ microseconds } (\mu s)$$
$$= 125 \times 10^{-6} s$$

$$\Rightarrow F_s = \frac{1}{125 \times 10^{-6}} = \frac{1000000}{125} = 8000 \text{ Hz} = 8 \text{ kHz}.$$



Q. Can we get the original signal back from the discrete-time signal? or

Q. What is the minimum sampling rate to acquire a complete reconstruction of the original signal from its sampled version?

→ Issue is aliasing!

To understand what aliasing is, consider the following example.

Consider two continuous-time signals:

$$x_1(t) = \cos(2\pi \cdot 10t)$$

$$x_2(t) = \cos(2\pi \cdot 50t)$$

that are sampled at a rate of $F_s = 40Hz$.

$$\Rightarrow T = \frac{1}{40} s$$

The discrete-time signals resulting from this sampling are:

$$x_1[n] = x_1(nT) = \cos\left(2\pi \cdot 10 \cdot \frac{n}{40}\right) = \cos\left(\frac{\pi}{2}n\right)$$

$$x_2[n] = x_2(nT) = \cos\left(2\pi \cdot 50 \frac{n}{40}\right) = \cos\left(\frac{5\pi}{2}n\right)$$

However, we note that

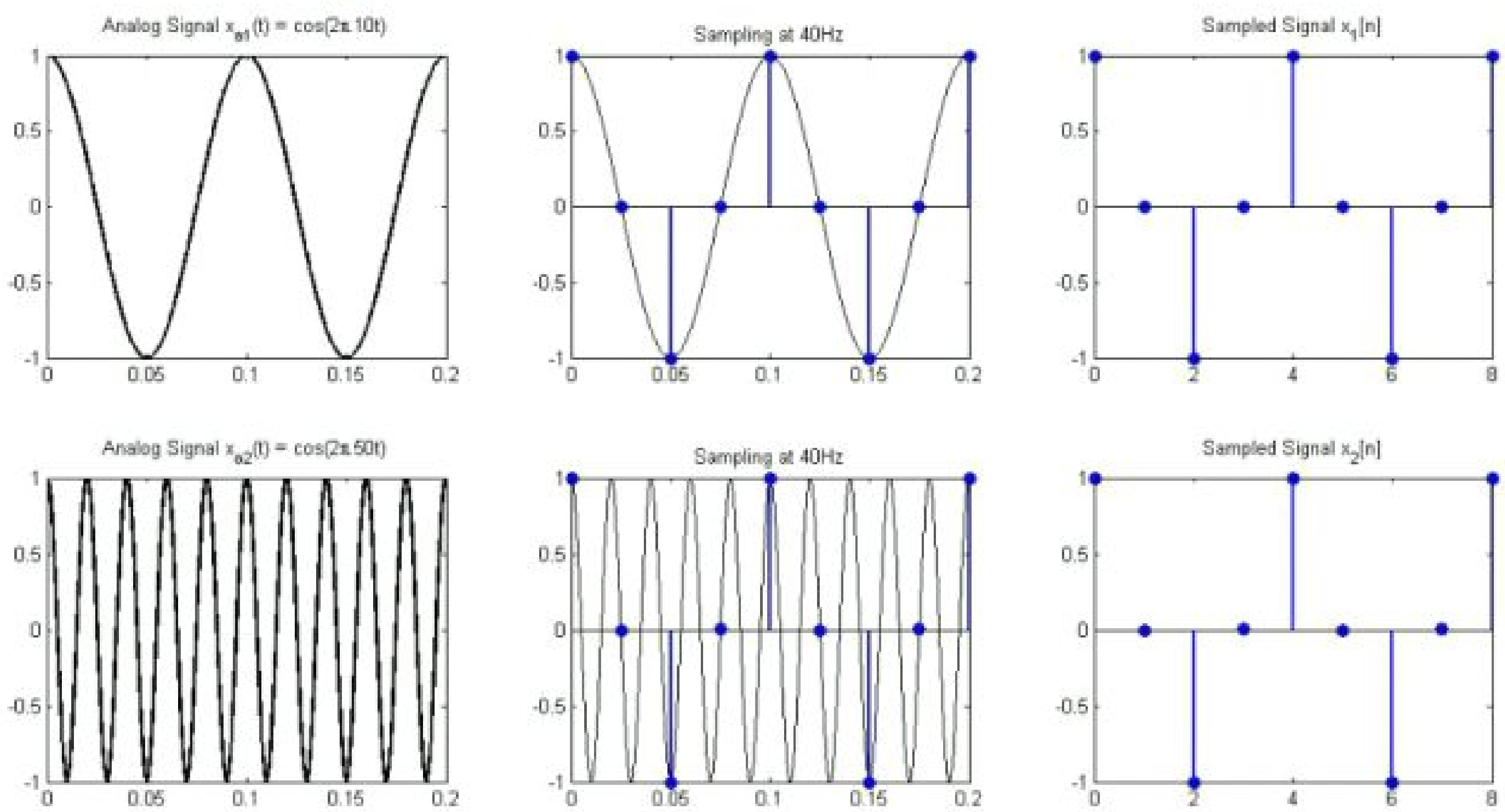
$$x_2[n] = \cos\left(\frac{5\pi}{2}n\right) = \cos\left(\left(2\pi + \frac{\pi}{2}\right)n\right) = \cos\left(\frac{\pi}{2}n\right) = x_1[n]$$

Therefore, the resulting sampled signals are identical. If we are given with sample values from $\cos(\frac{\pi}{2}n)$, there will be ambiguity as to whether these sample values are from $x_1[n]$ or $x_2[n]$ at a sampling rate of 40Hz.

The frequency (sinusoid frequency) $F_2 = 50Hz$ is an "alias" of the frequency $F_1 = 10Hz$ at the sampling rate of 40Hz.

In fact at a sampling rate of 40Hz, all of the sinusoids $\cos(2\pi \cdot (F_1 + K10)t)$, $K = 0, \pm 1, \pm 2, \cdots$ are aliases of $F_1 = 10Hz$.

Two different analog signals.



Resulting into the same sampled signal !!

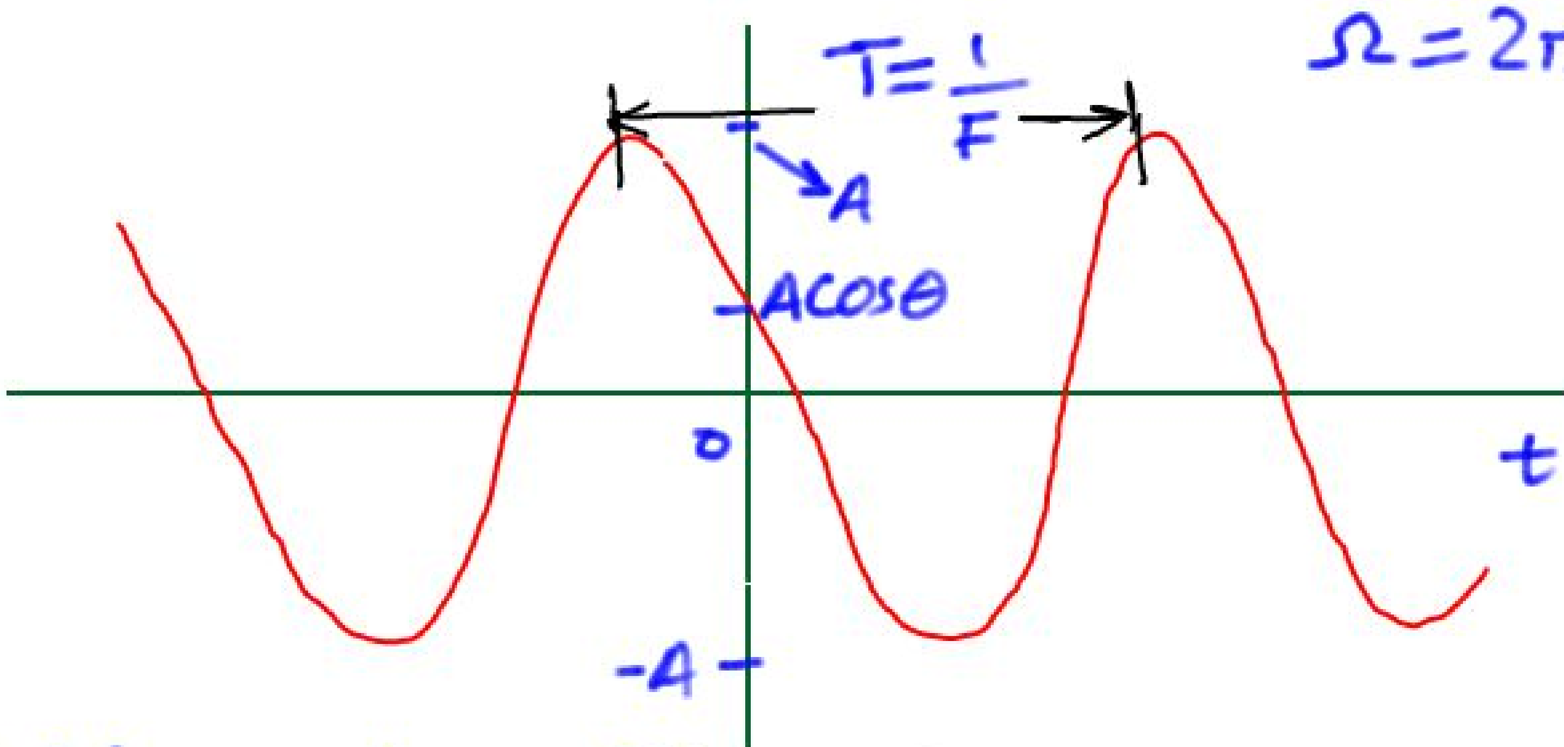
Comparison of Continuous-Time and Discrete-Time Sinusoidal Signals

A continuous-time sinusoidal signal is given by

$$x_a(t) = A \cos(\Omega t + \theta)$$

Analog Amplitude Angular Frequency Phase

$\Omega = 2\pi F$



Or it can be written as:

$$x_a(t) = A \cos(2\pi F t + \theta), \quad -\infty < t < \infty$$

We note that in continuous-time case,

1. For every fixed value of F, x_a is periodic:

$$x_a(t + T) = x_a(t)$$

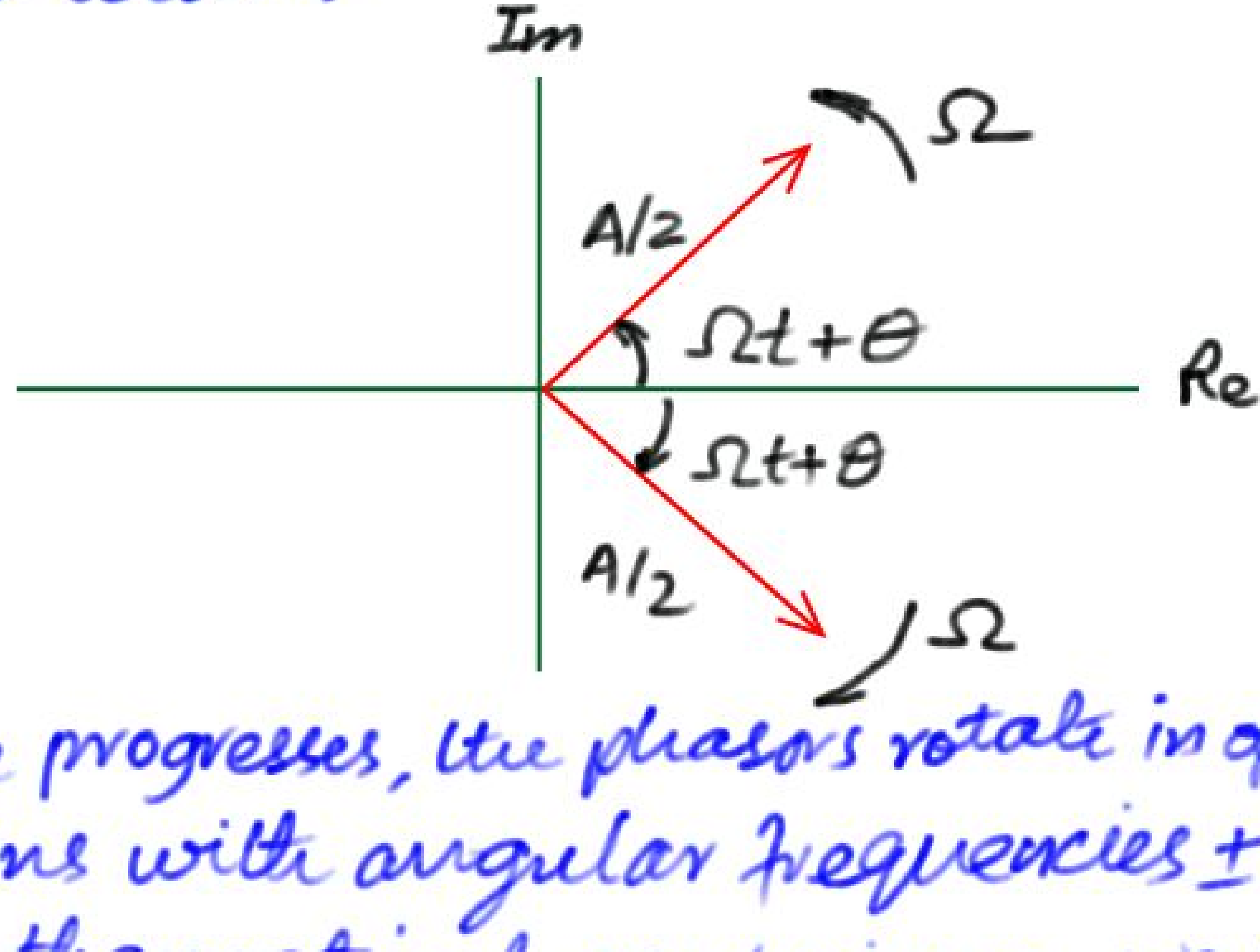
2. For every frequency F (or Ω), there is a unique and distinct sinusoidal signal.

3. Increasing the frequency F results in an increase in the rate of oscillations of the signal (i.e. more periods are included in a given time-interval).

By definition, 'frequency' F is an inherently 'positive' physical quantity (as it is obvious if we interpret 'frequency' as the 'number of cycles per second'. However, in many cases, for mathematical convenience, we have to introduce 'negative frequencies'. We can write

$$x_a(t) = A\cos(\Omega t+\theta) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$$

i.e. the sinusoid can be obtained by adding two complex-conjugate exponential signals (sometime called 'phasors') as shown below:



As time progresses, the phasors rotate in opposite directions with angular frequencies $\pm\Omega$. For mathematical convenience, we use both positive and negative frequencies i.e.

$$-\infty < F < +\infty \quad \text{or} \quad -\infty < \Omega < +\infty$$

For a discrete-time sinusoidal signal

$$x[n] = A\cos(\omega n+\theta), \quad -\infty<n<\infty$$

or $$x[n] = A\cos(2\pi f n+\theta) \text{ as } \omega=2\pi f$$

In this case we note that

1. A discrete-time sinusoidal signal is periodic only if its frequency f is a rational number.

Periodicity demands

$$x[n+N] = x[n] \text{ for all } n \text{ (period } N)$$

As $$\cos(\theta+k2\pi) = \cos\theta \text{ for } k=0,\pm1,\pm2,\cdots$$

$$\Rightarrow \cos(2\pi f_0(N+n)+\theta) = \cos(2\pi f_0 n+\theta)$$

$$\Rightarrow \quad 2\pi f_0 N = 2k\pi$$

$$\Rightarrow \quad f_0 = \frac{k}{N} \quad \text{(a rational number)}$$

2. Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.

As $$\cos[(\omega_0+2\pi)n+\theta] = \cos(\omega_0 n+\theta)$$

Therefore signals

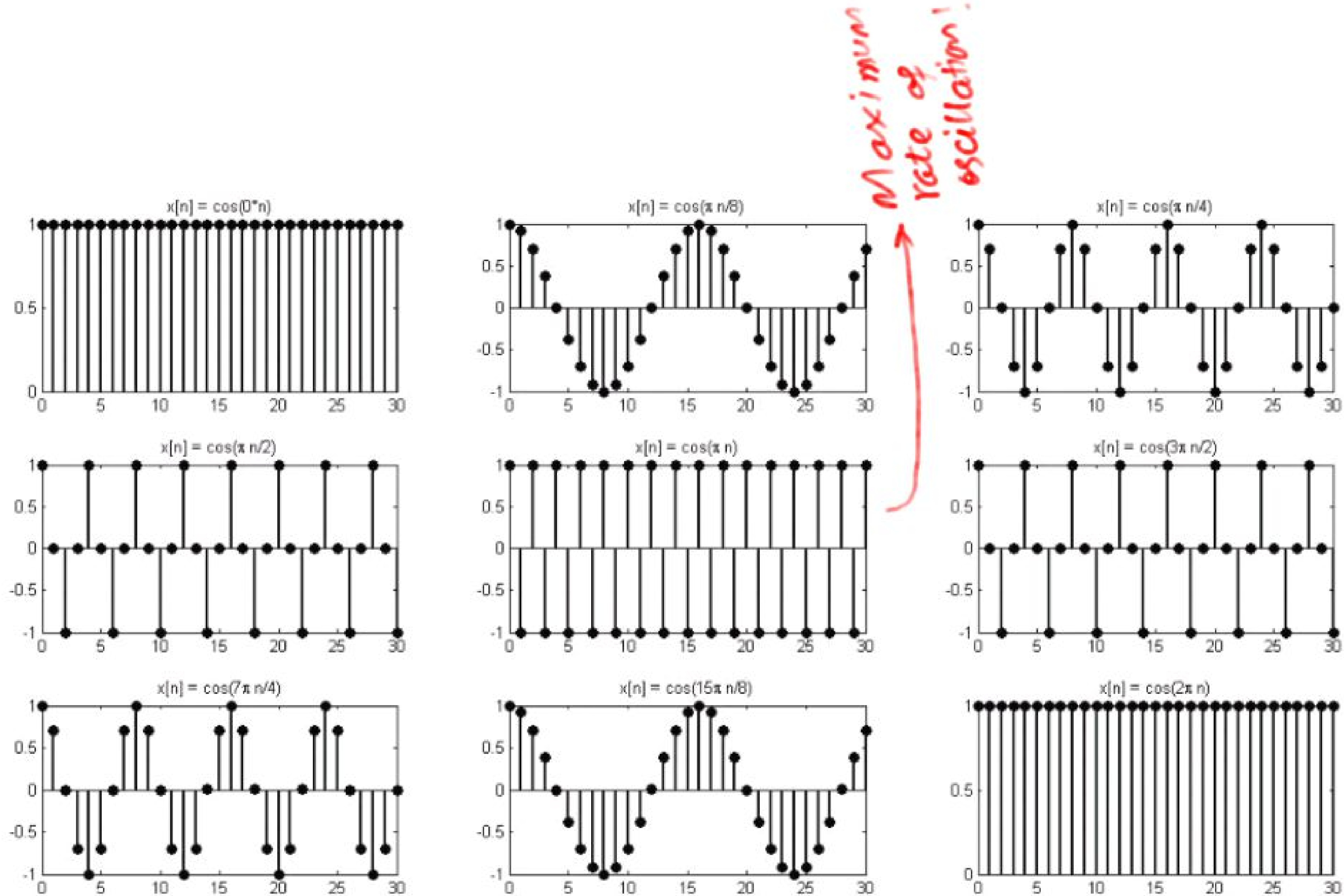
$$x_k[n] = A\cos(\omega_k n+\theta), \quad k=0,\pm1,\pm2\cdots$$

with $\omega_k = \omega_0 + k2\pi$ $\quad (-\pi<\omega_0<\pi)$

are indistinguishable. If frequencies are:

$$\left.\begin{array}{l} |\omega|<\pi \\ \text{or } |f|<\frac{1}{2} \end{array}\right\} \longrightarrow \text{unique signals}$$

otherwises aliases.



x[n] = cos(0*n)
x[n] = cos(π n/8)
x[n] = cos(π n/4)
x[n] = cos(π n/2)
x[n] = cos(π n)
x[n] = cos(3π n/2)
x[n] = cos(7π n/4)
x[n] = cos(15π n/8)
x[n] = cos(2π n)
Maximum rate of oscillation!

3. The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi$ (or $\omega = -\pi$) or equivalently when $f = \frac{1}{2}$ (or $f = -\frac{1}{2}$). This is illustrated with $x[n] = \cos(\omega n)$ with different ω on the previous page.

Sampling of Sinusoidal Signals

A further insight into the working of aliasing effects in sampling of continuous-time signals can be obtained by the study of sinusoidal signals.

Consider an analog sinusoidal signal

$$x_a(t) = \underbrace{A}_{\text{Amplitude}} \cos(2\pi \underbrace{F}_{\text{Frequency}} t + \underbrace{\theta}_{\text{Phase}})$$

If t is discretized as:

$$t = nT \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots \pm \infty$$

T being the sampling time related to the sampling rate by $f_s = 1/T$. The discrete-time sampled signal obtained in this way can be written as:

$$x[n] = x_a(nT)$$

$$\Rightarrow x[n] = A\cos(2\pi F n T + \theta)$$

$$= A\cos\left(2\pi \frac{F}{f_s} n + \theta\right)$$

Defining $f = F/f_s$ as the "relative" or "normalized" frequency, the sampled signal can be written as:

$$x[n] = A\cos(2\pi f n + \theta)$$

In case of a continuous-time sinusoidal signal,

$$-\infty < F < \infty \quad \text{or} \quad -\infty < \Omega < +\infty \qquad \text{— (1)}$$

The situation is different in case of discrete-time sinusoidal signals

$$-\frac{1}{2} < f < \frac{1}{2} \quad \text{or} \quad -\pi < \omega < \pi$$

$$\Rightarrow -\frac{1}{2} < \frac{F}{f_s} < \frac{1}{2} \quad \text{or} \quad \frac{-f_s}{2} < F < \frac{f_s}{2} \qquad \text{— (2)}$$

From relationships (1) and (2), it is clear that the fundamental difference between the continuous-time and discrete-time signals is in their range of values of the frequencies.

Sampling

$$\left\{ -\infty < F < \infty \ \text{or} \ -\infty < \Omega < \infty \right\} \xrightarrow{f_s} \left\{ -\frac{1}{2} < f < \frac{1}{2} \quad -\pi < \omega < \pi \right\}$$

Infinite Range $\qquad\qquad$ Finite range

The sampling of a continuous-time sinusoidal signal

$$x_a(t) = A\cos(2\pi F_0 t + \theta)$$

with a sampling rate $F_s = 1/T$, results in a discrete-time signal

$$x[n] = A\cos(2\pi f_0 n + \theta) \quad -(A)$$

where $f_0 = F_0/F_s$ is the relative or normalized frequency of the resulting sinusoid.

Case ① If we assume that

$$-F_s/2 \le F_0 \le F_s/2 \quad \text{(frequency range of analog signal)}$$

then the frequency f_0 of $x[n]$ is in the range

$$-\frac{1}{2} \le f_0 \le \frac{1}{2} \quad \text{(Frequency range of discrete-time signal)}$$

In this case, there is one-to-one relationship between F_0 and f_0 and hence it is possible to reconstruct the analog signal $x(t)$ from $x[n]$.

Case ② If the sinusoids

$$x_a(t) = A\cos(2\pi F_k t + \theta)$$

where

$$F_k = F_0 + kF_s, \qquad k = \pm 1, \pm 2, \pm 3, \cdots$$

are sampled at a rate F_s. It is clear that F_k is outside of the fundamental range of $-\frac{F_s}{2} \le f \le \frac{F_s}{2}$.

Consequently, the sampled signal is:

$$\begin{aligned} x[n] = x_a(nT) &= A\cos(2\pi F_k nT + \theta) \\ &= A\cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right) = A\cos\left(\frac{2\pi F_0}{F_s} n + \theta + 2\pi k n\right) \\ &= A\cos\left(2\pi \frac{F_0}{F_s} n + \theta\right) = A\cos(2\pi f_0 n + \theta) \quad -(B) \end{aligned}$$

which is identical to the discrete-time signal of equation (A).

The frequencies $F_k = F_0 + kF_s$, $-\infty < k < \infty$ (k being an integer) are indistinguishable from the frequency F_0 after sampling at a rate of F_s, and hence they are aliases of F_0.

The frequency $F_s/2$ (half of the sampling rate) is called the folding frequency.

As another example we consider

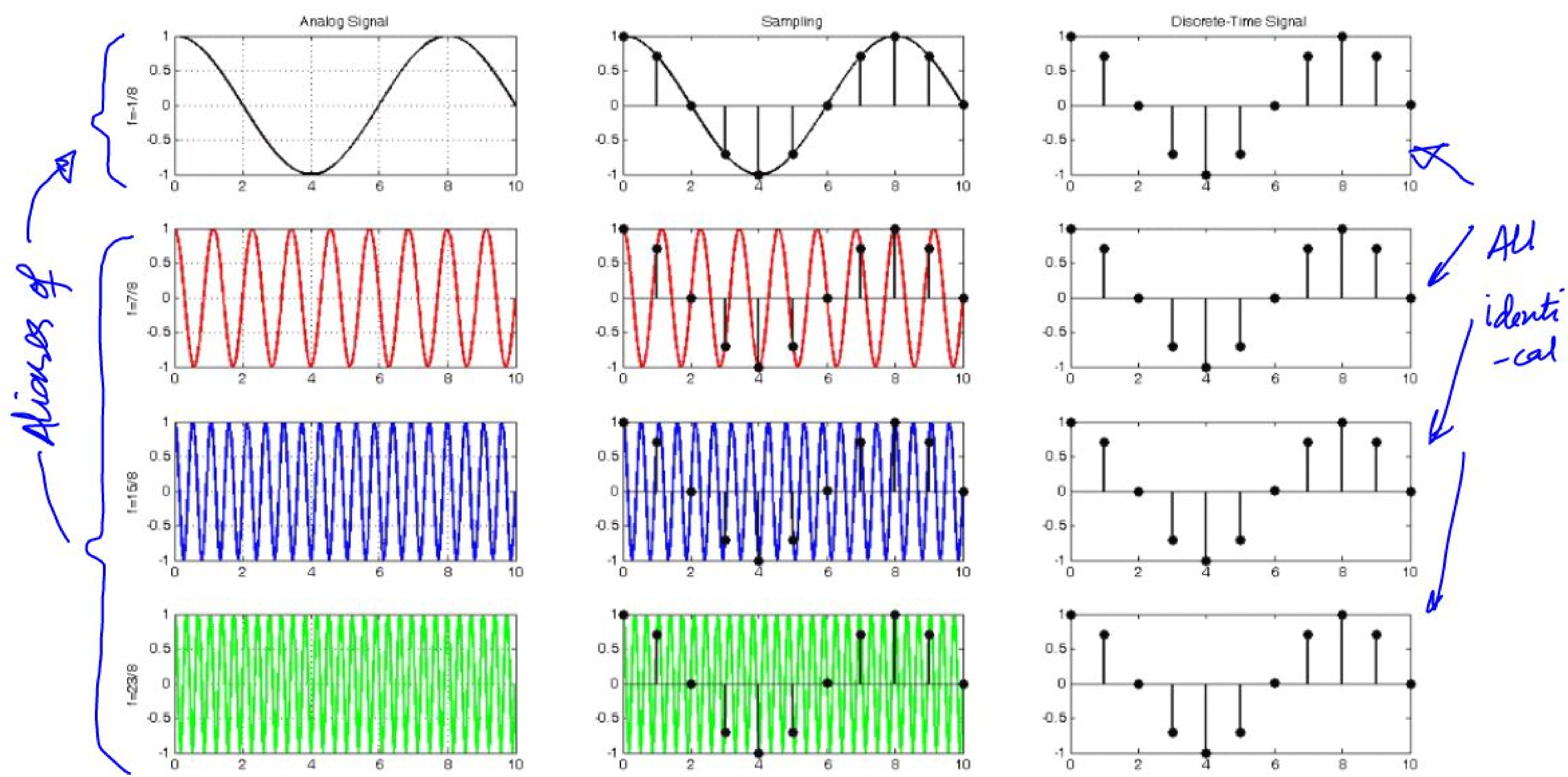
$$F_0 = -\frac{1}{8}\,\text{Hz} \Rightarrow x_0(t) = \cos(2\pi F_0 t)$$

$$F_1 = \frac{7}{8}\,\text{Hz} \Rightarrow x_1(t) = \cos(2\pi F_1 t)$$

$$F_2 = \frac{15}{8}\,\text{Hz}\ (= F_1 + 1) \Rightarrow x_2(t) = \cos(2\pi F_2 t)$$

$$F_3 = \frac{23}{8}\,\text{Hz}\ (= F_2 + 1) \Rightarrow x_3(t) = \cos(2\pi F_3 t)$$

All four analog sinusoids are sampled at a rate of $F_s = 1\,\text{Hz}$. The sampling process and the resulting discrete-time sampled signals are shown on the next page.


Analog Signal
Sampling
Discrete-Time Signal
f=-1/8
f=7/8
f=15/8
f=23/8
Aliases of
All identi-cal